THESE

PRESENTEE

A L'UNIVERSITE PIERRE ET MARIE CURIE

PARIS VI

POUR OBTENIR

LE GRADE DE DOCTEUR ES-SCIENCES PHYSIQUES

PAR

Vincent RIVASSEAU

"DEVELOPPEMENTS ASYMPTOTIQUES ET METHODES GRAPHIQUES EN PHYSIQUE MATHEMATIQUE"

Soutenue le 18 Juin 1982 devant la Commission d'Examen

MM. J.C LE GUILLOU — Président
E. BREZIN
C. DE CALAN
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UNIV. LEIDEN

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UNIV. LYON I

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Ayant appris au dernier moment que Messieurs les professeurs Jancovici et Nataf feraient partie de mon jury, je leur offre ici tous mes remerciements.

"Avant de te mettre à penser, réfléchis bien."
maxime polonaise (Stanislas Lec).

A mes parents,
A Marie-France, Jean-Noël
et ceux qui le suivront
je dédie ce travail en témoignage
d'amour et de reconnaissance.

Introduction Générale

La dernière décennie a vu se généraliser l'utilisation des méthodes euclidiennes en théorie des champs, à la suite de travaux jalonnés par les noms de Schwinger, Symanzik, Nelson, Guerra, Rosen, Simon, Glimm et Jaffe. Parallèlement, le fossé qui semblait séparer les deux "soeurs ennemies" de la physique mathématique, la mécanique statistique et la théorie des champs, ne cesse de se combler. Un point de vue largement répandu aujourd'hui identifie deux problèmes majeurs de ces disciplines: la "limite ultraviolette," ou le passage au continu, en théorie des champs, et l'approche du point critique pour un modèle correspondant sur un réseau, en mécanique statistique [1-5]. Dans sa diversité notre travail reflète un peu cette situation. (ainsi que les tendances éclectiques de l'auteur!), car des quatre parties qui le composent les deux premières sont consacrées à la théorie des champs et plus particulièrement à l'étude des amplitudes de Feynman, les deux suivantes traitent plutôt des questions de mécanique statistique. L'ensemble n'a guère d'unité et il n'a pas été facile de lui trouver un titre général.

On peut malgré tout dégager une idée directrice de cette thèse, celle d'étudier la nature des différents développements systématiques que l'on peut écrire pour les fonctions de Schwinger (ou de corrélation...) d'un modèle en interaction. Comme ces fonctions ne sont presque jamais explicitement

calculables, on a recours en général pour leur étude à des approximations obtenues en négligeant certaines quantités. En les systématisant on obtient pour les fonctions en questions divers développements. Chacune des parties de ce travail se rapporte plus ou moins directement à certains d'entre eux: au développement perturbatif renormalisé habituel dans la constante de couplage pour une théorie des champs lagrangienne (I et II), au développements dans les invariants ou les masses des amplitudes de Feynman (II), au développement à haute température d'un modèle sur réseau, qui peut prendre différentes formes (III et IV), enfin au développement en 1/N où N est le nombre de composantes des spins dans un modèle de Heisenberg (IV).

Ces développements sont souvent indexés par des graphes, et nous nous sommes vite aperçu que la théorie des graphes était la branche des mathématiques que nous utilisions le plus. Notre troisième partie est ainsi uniquement consacrée à des résultats de théorie des graphes qui en particulier démontrent une conjecture de E. Lieb, l'extension au modèle des rotateurs plans d'une inégalité de corrélation aux conséquences intéressantes, qu'il avait introduite et prouvée dans le cas du modèle d'Ising. Dans nos deux premières parties, l'étude des amplitudes de Feynman est inséparable de celle des fameux graphes ou diagrammes qui leur sont associés. Même les développements en chemins aléatoires que nous introduisons dans la partie IV

peuvent s'interpréter en terme de graphes, ce qui n'est pas surprenant puisqu 'ils correspondent en fait à un développement à haute température quelque peu déguisé.

Après ces remarques expliquant le titre finalement retenu pour l'ensemble de la thèse, revenons sur la nature des développements que nous y étudions. Ceux ci sont malheureusement en général divergents, et nous fournissons d'ailleurs un nouveau résultat rigoureux à ce sujet (I.4). La question se pose donc de leur rapport mathématique précis avec les quantités qu'ils sont censés représenter. Le rapport minimal qu'on est en droit d'espérer, c'est l'asymptoticité qui est pourtant rarement évidente. Cette notion existe pour n'importe quel développement selon une échelle de fonctions. Dans le cas des amplitudes de Feynman, lorsque l'on dilate certains invariants et certaines masses par un paramètre λ très grand, l'échelle naturelle est celle des puissances et puissances de logarithmes de λ. M. Bergère, Y. Lam, C. de Calan et A. Malbouisson [6-8] ont montré comment écrire dans cette échelle un développement asymptotique pour n'importequelle amplitude de Feynman renormalisée dans des conditions de dilatation trés générales. Ce développement s'obtient de facon particulièrement transparente dans la représentation des amplitudes dite de Mellin Complet, à laquelle est consacrée notre deuxième partie [9-10].

Toutefois on ne peut pas considérer qu'un développement simplement asymptotique à une certaine quantité contient à

lui seul toute l'information relative à celle ci. Il faut en plus pour cela qu'il soit possible de reconstruire de manière unique cette quantité à partir des termes du développement, par un procédé bien défini. Dans le cas d'un développement en série entière divergent le point autour duquel on développe ne peut bien sûr pas être un point d'analyticité; mais s'il se trouve au bord d'un domaine d'analyticité et si la série de Taylor est asymptotique de manière assez forte et uniforme à la fonction analytique en question (cas de la quasi-analyticité) il existe de tels procédés de reconstruction de la fonction à partir de cette série. L'un d'entre eux, le procédé de sommation de Borel [11-16], s'est révélé assez souvent applicable aux séries entières rencontrées en physique mathématique. Les références [11-28] donnent une liste non limitative de telles applications rigoureuses. Nous ajoutons, dans la partie IV, un item à cette liste en montrant que le développement en 1/N du modèle O(N) σ. non linéaire sur réseau est Borel sommable à haute température. Mais le procédé de resommation de Borel, combiné avec des estimations des grands ordres des séries considérées, a été appliqué aussi de manière non rigoureuse, avec des résultats numériques précis, physiquement plus intéréssants ([29-39], liste encore moins limitative que la précédente!). C'est une tâche mathématique de longue haleine que d'essayer de rendre rigoureux cette masse considérable de résultats et les premiers pas dans cette voie n'ont été accomplis

que récemment [40-43,76]. La première partie de cette thèse se rattache à cet ambitieux programme, aussi nos résultats y sont plus partiels. Nous définissons rigoureusement dans un voisinage de l'origine la transformée de Borel du développement perturbatif renormalisé pour le modèle ϕ^4 de théorie des champs en dimension ν complexe pour $Re\nu<4$, et en dimension 4. Jusqu'à présent on ne sait pas construire des fonctions de Schwinger satisfaisantes pour ces modèles et l'on ne dispose donc que du développement perturbatif, excepté aux points $\nu = 1,2,3$, où l'on sait que celui ci est Borel sommable. Les autres cas correspondent à des modèles au statut encore incertain. Nous pensons avec A. Wightman que l'étude de la transformée de Borel du développement perturbatif peut se révéler importante pour les comprendre. Bien que cette voie semble semée de difficultés décourageantes, il n'est tout d'abord pas exclu à priori que l'on puisse obtenir la continuation analytique dans un voisinage du demi axe positif des germes de fonctions que nous avons définis, avec de bonnes conditions de croissance à l'infini. Ceci permettrait d'obtenir directement la construction des fonctions de Schwinger correspondant au développement perturbatif, par application des réciproques des théorèmes de Watson ou de Nevanlinna [13,14], par exemple dans le cas de ϕ^4_ν, $Re\nu<4$, qui est une théorie super-renormalisable [I-]. Un objectif moins ambitieux est le calcul du rayon de convergence exact de la transformée de Borel, qui a été mené à bien dans le cas de l'oscillateur anharmonique et

de ϕ_2^4 [40,42,76], et la localisation de la première singularité, qu'on s'attend à rencontrer sur l'axe réél négatif pour $\nu = 3$ (I,4). Pour $\nu = 4$, on pense plutôt la trouver sur l'axe réél positif: c'est la conjecture de t'Hooft-Parisi [36,38], reliée aux idées de Landau, Pomeranchuk et collaborateurs, et Feldman [44-46]. Une démonstration de l'existence d'une telle singularité, qu'on l'appelle "renormalon" ou "fantôme de Landau" nous semblerait un résultat mathématique tout à fait impressionant, qui éclairerait sous un jour différent et complémentaire les tentatives de "théorie destructive des champs" visant à prouver que la limite continue d'une théorie ϕ_4^4 sur réseau est nécessairement un champ libre [5,47,48]. Il s'agit là d'un des axes de recherche selon lesquels peut se poursuivre notre travail.

L'organisation de la thèse est la suivante: chaque partie contient la reproduction d'un ou plusieurs articles, et souvent une introduction, des rappels ou des commentaires visant à mieux les situer dans leur contexte et à rendre l'ensemble plus autonome. Les Références de tous ces passages ainsi que de la présente introduction sont groupées à la fin de la thèse.

PREMIERE PARTIE

Etude de l'interaction ϕ^4 en théorie des champs

"Il me semble inopportun d'essayer d'améliorer la rigueur des preuves de Pomeranchuk, en particulier parce que la briéveté de la vie ne nous permet pas le luxe de dépenser du temps sur des problèmes qui n'amèneront pas de résultats nouveaux"

L. Landau

Non Perturbative Dimensional Interpolation*

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ABSTRACT

We propose a program for constructing non-perturbative dimensional interpolation of the ϕ^4 model in the super-renormalizable region $\mathrm{Re}\ \nu < 4$. As a first step in this direction, we deduce the local existence of the Borel transform of the interpolated perturbative series of the model, near the origin, from a very simple exponential bound on contributions of individual diagrams in the region $\mathrm{Re}\ \nu < \frac{10}{3}$.

*This paper is partly an elaboration of a talk by the second-named author at the Strasbourg Meeting, June 1978 (RCP 25 of the CNRS). It contains also some results of the first-named author's third cycle thesis at the University of Paris VI. Both authors thank the Centre de Physique Théorique of the Ecole Polytechnique for hospitality during 1977-1978.

**On leave from Centre de Physique, Ecole Polytechnique

Interpolation Dimensionelle Non-perturbative

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Au cours de ces dernières années, la théorie constructive des champs relativistes quantifiés a atteint l'un de ses objectifs fondamentaux. On dispose maintenant d'une famille de modèles théoriques, dans l'espace temps de dimension deux et trois, dont les prévisions sont raisonnables du point de vue physique. Nous ne voudrions pas dire que tous les problèmes concernant ces modèles sont résolus; par example il subsiste toujours celui de la "complétude asymptotique". Néanmoins nous pensons qu'aujourd'hui, sans aucun doute, les rêves des pionniers de la théorie quantique des champs sont réalisés par ces modèles, super-renormalisables; les sceptiques ont battu en retraite car ces théories décrivent des champs relativistes vraiment en interaction.

Le problème fondamental s'est déplacé; est ce que les théories renormalisables mais non super-renormalisables ont aussi des solutions intéréssantes? La contribution présente n'apporte pas de réponse à cette question. Nous avons plutôt tenté de mieux comprendre la différence entre les théories super-renormalisables et strictement renormalisables, en étudiant un modèle ϕ^4_ν , comme fonction de la dimension, ν . Ce n'est pas une idée nouvelle, mais elle n'a pas encore été développée dans le cadre de la théorie constructive des champs.

On sait bien, par contre, interpoler et renormaliser dimensionelle-

ment les termes de la série perturbative du modèle, comme nous le rappelerons. Notre projet est de prendre ces résultats bien établis pour point de départ afin de construire les quantités fondamentales de la théorie, les fonctions de Schwinger, par sommation de Borel. On étudiera la théorie euclidienne comme il est d'usage désormais. Dans le présent article nous nous contentons d'expliquer la méthode proposée et de démontrer un premier résultat , à savoir l'existence et l'analyticité de la transformée de Borel de la série des perturbations dans un disque autour de l'origine, pour Re $\nu < \frac{10}{3}$, ν désignant la dimension d'espace temps. Nous espérons étendre bientôt ce résultat à tout le domaine superrenormalisable, Re $\nu < 4$. Nous commençons par une revue des points de vue généralement admis sur ce sujet mais non rigoureusement démontrés, puis des résultats rigoureux de la théorie constructive; le lecteur connaissant ces questions peut passer directement au paragraphe III.

I. La "sagesse conventionnelle" concernant le modèle $\lambda\phi^4_\nu$ comme fonction de ν

Nous allons considérer deux sortes de quantités attachées au modèle: les fonctions de Schwinger $S^{(k)}$ $k = 0,1,2...$ dont l'expression formelle dans un espace-temps de dimension ν est:

$$S^{(k)}(x_1...x_k) = \lim_{|\Lambda|\to\infty} Z(\Lambda)^{-1} \int D\phi \prod_{j=1}^{k} \phi(x_j) \exp - \int_\Lambda \{((\partial\phi)^2 + m_0^2\phi^2) + \lambda\phi^4\}d^\nu x \tag{1}$$

$$Z(\Lambda) = \int \phi \exp - \int_\Lambda \{\tfrac{1}{2}(\partial\phi^2 + m_0^2\,\phi^2) + \lambda\phi^4\}d^\nu x \tag{2}$$

et la pression, p , donnée par:

$$p = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \ell n\, Z(\Lambda) \tag{3}$$

où Λ est mesurable borné, et $|\Lambda|$ son volume.

Ce n'est qu'une expression formelle parce qu'on doit d'abord ajouter à l'éxposant des contretermes nécessaires pour la renormalisation; et même alors des difficultés subsistent pour montrer que les intégrales sont bien définies et que la limite thermodynamique $|\Lambda| \to \infty$ peut être effectuée.

De ces fonctions de Schwinger on passe aux fonctions de Schwinger tronquées $S^{(n)T}$. Par exemple:

$$S^{(4)T}(x_1,x_2,x_3,x_4) = S^{(4)}(x_1,x_2,x_3,x_4) - S^{(2)}(x_1,x_2)S^{(2)}(x_3,x_4) - S^{(2)}(x_1,x_3)S^{(2)}(x_2,x_4) - S^{(2)}(x_1,x_4)S^{(2)}(x_2,x_3) \tag{4}$$

Plus généralement $S^{(n)T}$ est définie par l'éxpression:

$$S^{(n)T}(x_1,\ldots x_n) = \sum_{\substack{\text{toutes les partitions} \\ A_1,\ldots A_p \text{ de } [1,\ldots n]}} (-1)^{p+1} \prod_{k=1}^{p} S^{|Ak|}(x_{k1},\ldots,x_{k|A_k|}) \qquad (5)$$

où $A_k = \{k1,\ldots k|A_k|\}$. Pour l'interaction ϕ^4 et les solutions symétriques selon $\phi \to -\phi$, les fonctions de Schwinger d'ordre impair sont nulles ce qui explique la formule (4).

La transformée de Fourier de $S^{(4)T}$ est de la forme

$$S^{(4)T}(p_1,p_2,p_3,p_4) = \delta(\sum_{j=1}^{4} p_j) \prod_{j=1}^{4} \hat{S}^{(2)}(p_j)\ T^{(4)}(p_1,p_2,p_3,p_4)$$

et la constante de couplage renormalisée ordinaire est définie par:

$$g = -m^{-2}\ \Gamma^{(4)}(p_1^{(0)},\ldots p_4^{(0)}) \qquad (6)$$

$p_1^{(0)},\ldots p_4^{(0)}$ est un point pris par convention, en général à l'origine $p_1^{(0)} = \ldots = p_4^{(0)} = 0$, ou au point, appelé point symétrique, où $p_j^0\ p_k^0 = \frac{m^2}{3}\ (4\ d_{jk} - 1)$; m est la masse physique d'une particule. Evidemment on a besoin d'informations théoriques sur les solutions du modèle pour savoir si ces quantités sont toutes bien définies. En fait c'est en terme de séries formelles en puissances de g que la théorie traditionnelle des perturbations est exprimée. Le coefficient de g^n est une somme de contributions étiquetées par les diagrammes de Feynman.

Une telle contribution est fourmie par une intégrale pouvant présenter des divergences quand deux arguments coïncident dans l'espace-temps: $x \sim y$ (divergence ultraviolette) ou quand certains arguments tendent vers l'infini (divergence infra-rouge). Le processus de renormalisation examine l'intégration sur tous les ensembles possibles de variables associées à un sous diagramme; lorsqu'une divergence apparaît pour un tel sous diagramme

la renormalisation la rend finie par l'addition d'un contreterme. Deux points sont donc importants; choisir judicieusement les contretermes pour obtenir un résultat fini, mais aussi respecter la structure d'un graphe avec tous ses sous graphes, lorsqu'on soustrait les contretermes, la solution étant la fameuse méthode de soustraction de Bogoliubov, Parasiuk-Hepp-Zimmermann.

Nous voyons apparaître une premiere grande classification: s'il n'existe qu'un nombre fini de diagrammes de Feynman divergents, tels qu'après leur renormalisation partout où ils apparaissent, tout diagramme de Feynman devient convergent, alors le modèle est dit superrenormalisable. Si non, le modèle est strictement renormalisable, ou peut être non-renormalisable, s'il n'y a pas d'entier ℓ tel que les diagrammes à plus de ℓ "jambes" soient convergents une fois les sous diagrammes à moins de ℓ "jambes" renormalisés.

Le modèle $\lambda\phi^4_\nu$ est superrenormalisable à $\nu = 2$ et 3.

En dimension 2, les seuls diagrammes dont il faut renormaliser la contribution sont:

pour les fonctions de Schwinger (7)

pour la pression (8)

après quoi toute les contributions sont finies.

En dimension 3, il faut renormaliser

pour les fonctions de Schwinger (9)

pour la pression (10)

En dimension 4, les contributions des diagrammes précédents sont divergentes mais par surcroît:

(11)

et une infinité d'autres à deux jambes.

Pour la pression, les contributions de (10) sont divergentes mais aussi:

et une infinité d'autres (12)

Les diagrammes à quatre jambes sont aussi divergents, à l'exception du diagramme trivial :

et une infinité d'autres (13)

A partir de six jambes, on n'a plus de divergences après la renormalisation des sous diagrammes divergents, de quatre jambes ou moins. Donc $\lambda\phi_4^4$ est renormalisable, mais pas superrenormalisable.

Passons maintenant a la théorie du groupe de renormalisation. Le résultat le plus important pour notre discussion est une famille de relations entre

$$\hat{S}^{(k)T}(\sigma p_1,\ldots,\sigma p_n;g,m) \qquad \sigma > 0 \tag{14}$$

et $$\hat{S}^{(k)T}(p_1,\ldots,p_n;g(\sigma),m(\sigma))$$

Les fonctions $\sigma \to g(\sigma)$ et $\sigma \to m(\sigma)$ fournissent la masse et la constante de couplage effectives. Le comportement asymptotique de $\hat{S}^{(k)T}(\sigma p_1,\ldots\sigma p_n;g,m)$ quand σ tend vers 0 ou $+\infty$ est donc déterminé

par celui des fonctions $g(\sigma)$ et $m(\sigma)$, à son tour gouverné par la célèbre fonction β .

On peut conjecturer le comportement suivant pour la fonction $g \to \beta(g)$ des modèles ϕ^4_2 ou ϕ^4_3 :

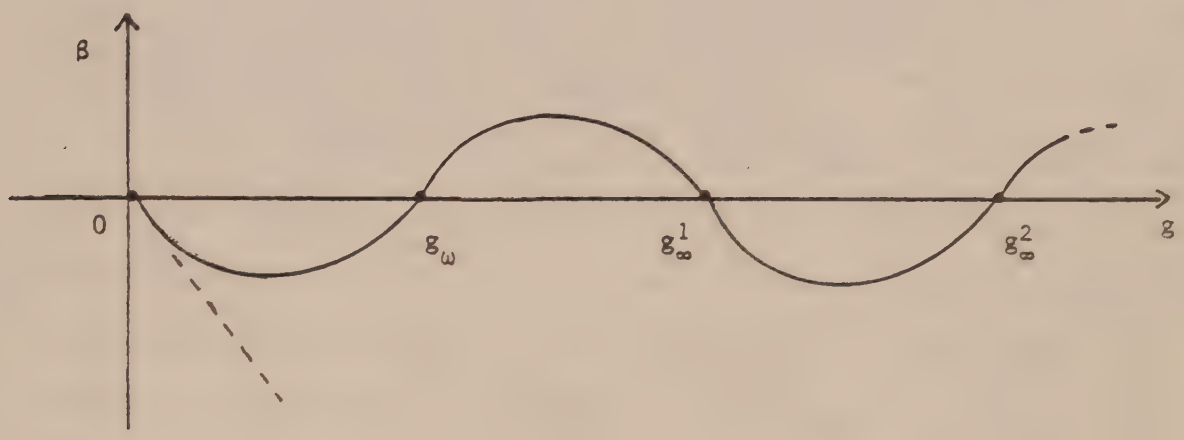


Figure 1. Fonction $\beta(g)$

L'existence de zéros tels que $\hat{g}^1_\infty$, g^2_∞ est encore une hypothèse, à l'appui de laquelle on peut citer les calculs numériques récents de N. Khuri [1]. Un zéro, g_0 , de β est attracteur infra-rouge ou ultra-violet selon que $\beta'(g_0)$ est positif ou négatif. A chaque intervalle entre deux zéros correspond une fonction $g(\sigma)$ dont la limite lorsque σ tend vers 0 (resp. $+\infty$) est l'attracteur infra-rouge (resp. ultra-violet). La théorie constructive prouve l'existence de $\beta(g)$ près de l'origine; son développement perturbatif commençant avec un terme linéaire négatif. β se comporte comme indiqué sur la figure près de l'origine. A notre connaissance l'existence de $\beta(g)$ n'a pas été

rigoureusement établie plus loin jusqu'à présent, mais on a néanmoins de très bonnes raisons de croire le dessin qualitativement exact dans l'intervalle $[0,g_\omega]$ avec $g_\omega < \infty$. [2] Le point g_ω est attracteur infra-rouge, l'origine attracteur ultra-violet. Pour les théories existant éventuellement dans l'intervalle $g_\omega < g < g^1_\infty$ le comportement ultra-violet est déterminé par le point g^1_∞ et ainsi de suite.

Fischer et Wilson ont eu l'idée remarquable d'interpoler la dimension de l'espace temps, et d'étudier des séries formelles en $(4-\nu)$ pour calculer la fonction β . Leur résultat apparaît dans la figure suivante:

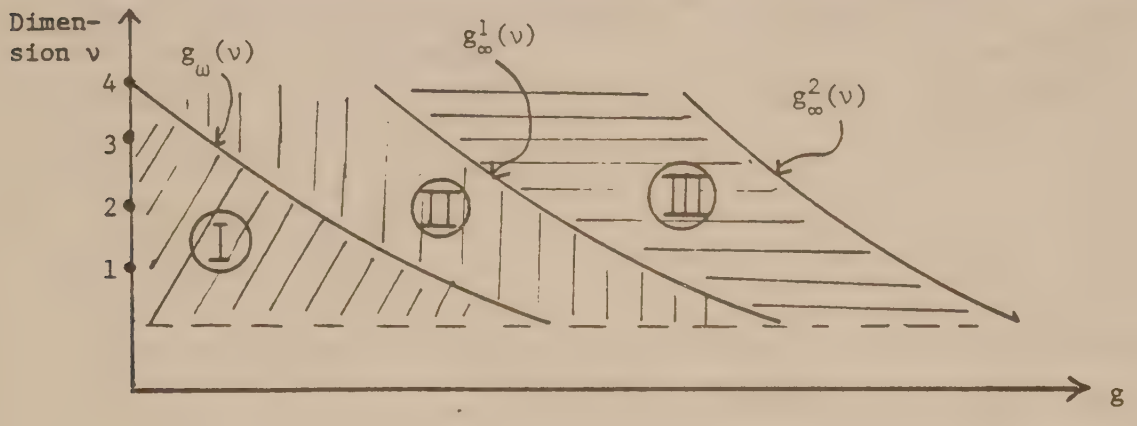


Figure 2: Comportement conjecturé des fonctions $g_\omega(\nu)$, $g^1_\infty(\nu)$,... avec une masse physique fixée

On a des séries formelles pour $g_\omega(\nu)$ près de $\nu = 4$ indiquant le comportement représenté, qui réduit le domaine (I), des théories super-renormalisables, à la théorie libre à $D = 4$. Il semble que ce soit le domaine naturel maximal pour étendre les méthodes de la théorie constructive à $D = 2$ et 3: on peut donc penser trouver une famille de fonctions de Schwinger analytiques en g et ν dans ce domaine, coïncidant avec celles des théories déja construites à $D = 2$ et 3 . Allant plus loin on peut

penser unifier également les théories dans la région II , ou la région III ; dans les deux cas le comportement ultra-violet est gouverné par g_∞^1 . Comme à $D = 4$ la théorie est strictement renormalisable on s'attend à la même chose pour $g_\omega(\nu) < g < g_\infty^2(\nu)$ et $\nu < 4$. Pour $g > g_\infty^2(\nu)$ on ne sait pas ce que l'on doit conjecturer; peut être des théories non-renormalisables dont les fonctions de Schwinger sont trop singulières pour être des distributions tempérées.

Si la vue d'ensemble ainsi esquissée est correcte il y a une sorte de transition de phase ultra-violette pour $g = g_\omega(\nu)$. A gauche les singularités ultra-violettes des fonctions de Schwinger sont superrenormalisables et déterminées par celles du champ libre. A droite elles seraient strictement renormalisables et déterminées par le comportement de la théorie avec constante de couplage $g_\infty^1(\nu)$. Jusqu'ici une telle transition de phase ultra-violette a été seulement étudiée en théorie constructive des champs, sur la frontière $g = g_\omega$ par Glimm et Jaffe [3]. C'est un problème capital ouvert, d'établir l'existence de la solution pour $g > g_\omega$.

Ici nous nous occupons d'un problème plus simple dont la solution néanmoins donnerait un appui sérieux à ces conceptions: construire une interpolation non-perturbative en dimension ν des solutions superrenormalisables.

Rappelons maintenant l'information fournie par la théorie constructive des champs sur le modèle ϕ^4 .

II. Résultats de théorie constructive des champs

Des solutions de $\lambda\phi_1^4$, $\lambda\phi_2^4$, $\lambda\phi_3^4$ ont été construites par des méthodes non-perturbatives. La renormalisation est effectuée par l'addition de contretermes à l'Hamiltonien, qui sont ceux indiqués par la théorie perturbative. En dimension 1, aucun contreterme n'est nécessaire; en dimension 2 il suffit de passer à l'ordre de Wick dans l'interaction: formellement $\lambda\phi^4 \to \lambda{:}\phi^4{:} = \lambda[\phi^4 - 6\langle\phi^2\rangle\phi^2 + 3\langle\phi^4\rangle]$. En dimension 3 on remplace $\lambda\phi^4$ par $\lambda{:}\phi^4{:}$ mais il faut en plus renormaliser les contributions des diagrammes divergents de (9) et (10)

L'interaction avec cutoff, employée par Magnen et Sénéor [4] s'écrit par exemple:

$$\mathcal{L}_{\pm}(\Lambda,\lambda,\kappa) = \lambda \int {:}\phi^4(x){:}\Lambda(x)d^3x + \frac{\lambda^2}{2}\int \left(\int {:}\phi^4(x){:}\Lambda(x)d^3x\right) d\mu_{0,\kappa}(\phi)$$
$$- \frac{\lambda^3}{6}\int \left(\int {:}\phi^4(x){:}\Lambda(x)d^3x\right)^3 d\mu_{0,\kappa}(\phi) - \frac{\lambda^2}{2}\delta m^2 \int {:}\phi^2{:}(x)\Lambda(x)\, d^3x \qquad (15)$$

où ϕ est une variable aléatoire gaussienne de moyenne zéro et covariance:

$$C_\kappa(x,y) = \frac{1}{(2\pi)^3}\int \frac{e^{ik\cdot(x-y)}}{k^2 + m^2}\,\eta_\kappa(k)\, d^3k$$

et η_κ une "fonction de cutoff"

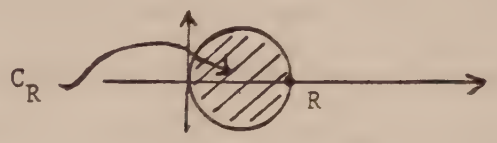
$$\eta_\kappa(k) = \eta\left(\frac{k}{\kappa}\right) \quad , \quad \text{où } \eta \text{ , fonction } C^\infty \text{ vaut } 0$$

pour $|k| > 2$ et 1 pour $|k| < 1$. $d\mu_{0,\kappa}$ est la mesure gaussienne associée a C_κ . Dans la construction de la solution on prend la limite ultra-violette $\kappa \to \infty$, puis thermodynamique, $\Lambda \to 1$. Le deuxième et le troisième terme dans (15) correspondent à et : le quatrième, à .

Le paramètre λ n'étant pas renormalisé, il est naturel de considérer

un développement perturbatif des fonctions de Schwinger en λ, constante de couplage nue. Dimock a démontré que les fonctions de Schwinger de $\lambda:\phi^4:_2$ sont infiniment différentiables en λ dans un intervalle $0 \leq \lambda < \varepsilon$ pour m_0 masse nue, strictement positive. La série de Taylor a l'origine $\lambda = 0$, développement perturbatif habituel, est donc asymptotique à la solution de la théorie constructive. La détermine t'elle uniquement? Une réponse positive à cette question est la méthode de sommation de Borel, utilisant par exemple le théorème suivant (Nevanlinna, Sokal [5]).

<u>Théorème</u> Soit f analytique dans le disque $C_R = \{z/\mathrm{Re}\,\frac{1}{2} > \frac{1}{R}\}$ tangent à l'origine:

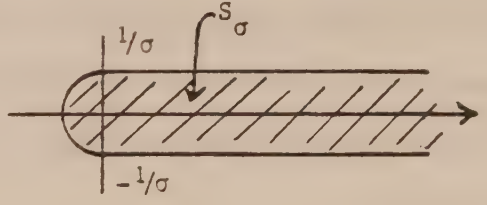


Si l'on a une série entière asymptotique à f , $\Sigma a_k z^k$, posons

$$R_N(z) = f(z) - \sum_{k=0}^{N} a_k z^k \quad .$$

Si
$$|R_N(z)| \leq A\sigma^N N! |z|^N \tag{17}$$

uniformément en N et $z \in C_R$, alors la transformée de Borel de f , $B(t) = \Sigma a_n \frac{t^n}{n!}$ converge pour $|t| < \frac{1}{\sigma}$, a une continuation analytique dans la bande $S_\sigma = \{t/\mathrm{dist}(t,R_+) < \frac{1}{\sigma}\}$:



satisfaisant la borne

$$|B(t)| \leq K \exp\left(\frac{|t|}{R}\right) \tag{18}$$

On a pour f la representation intégrale absolument convergente:

$$f(z) = \frac{1}{2}\int_0^\infty e^{-t/z}\, B(t)dt \ , \ z \in C_R \qquad (19)$$

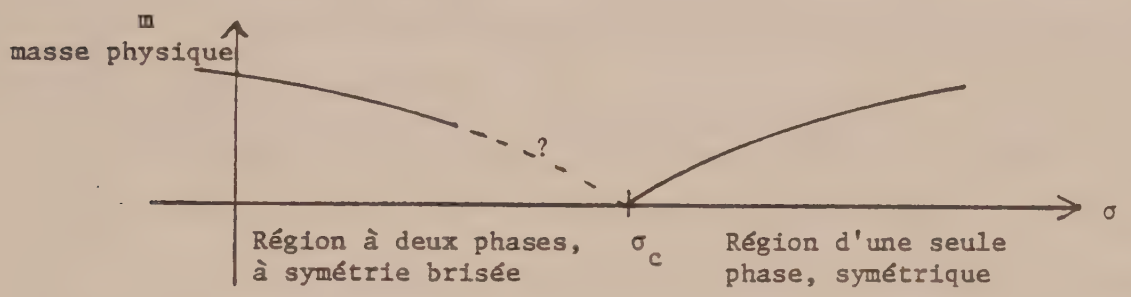
Réciproquement si $B(t)$ est analytique dans $S_{\sigma'}$ avec $\sigma' < \sigma$ et satisfait la borne (18) alors f définie par (19) est analytique dans C_R et satisfait la borne (17) uniformément dans $C_{R'}$, $R' < R$, si l'on pose $a_n = B^{(n)}(t)\Big|_{t=0}$.

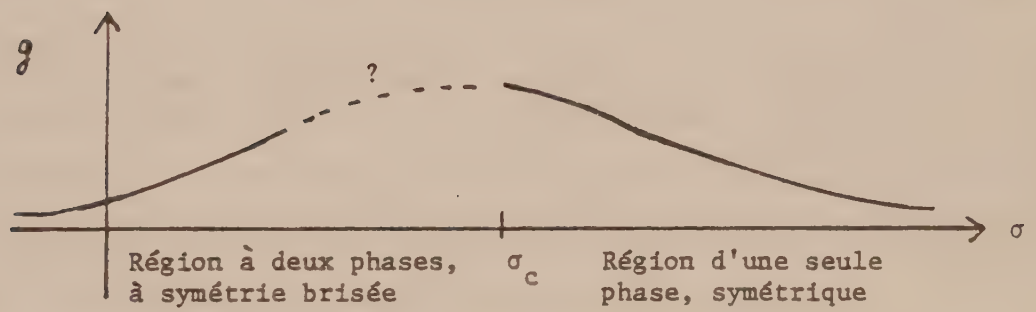
Par abus de langage on dira aussi que B est la transformée de Borel de la série $\Sigma\, a_k z^k$.

Pour $\lambda\phi_1^4$, Graffi, Grecchi et Simon ont établi que la pression est sommable de Borel. Il semble qu'une démonstration pour les fonctions de Schwinger n'est pas parue mais personne ne doute de la validité de l'assertion. Pour $\lambda\phi_2^4$ et $\lambda\phi_3^4$, Magnen et Sénéor ont établi la sommabilité de Borel pour la pression et les fonctions de Schwinger. Leur résultat, un des sommets de la théorie constructive, suggère l'approche suivante au problème d'interpolation dimensionnelle: Les coefficients de Taylor en $\lambda = 0$ des fonctions de Schwinger et de la pression sont interpolables en dimension par les processus, bien connus désormais, de la théorie de la renormalisation dimensionnelle. S'ils fournissent aussi une transformée de Borel holomorphe dans une région S_σ et telle que l'intégrale (19) converge, on peut prendre cette intégrale pour definition d'une interpolation dimensionnelle non-perturbative du modèle. On pourrait laisser la question des séries perturbatives dans la constante de couplage renormalisée, g , à une étape ultérieure.

Mentionnons que la théorie constructive établit l'existence de solutions possédant la symétrie $\phi \to -\phi$ et de solutions avec cette symétrie brisée. Pour avoir une vue d'ensemble il convient d'étudier l'effet d'une

perturbation de la forme $\frac{\sigma}{2}\phi^2$. On montre alors que pour σ assez grand la masse physique est fonction monotone croissante différentiable de σ , et qu'elle s'annule en un point σ_c . Le comportement qualitatif des paramètres physiques est donc:





Pour g petit on a donc deux types de solutions. Nous nous intéresserons désormais à la solution symétrique $(\sigma > \sigma_c)$. On sait que $g(\sigma)$ est bornée pour $\sigma > \sigma_c$, mais il reste à établir plusieurs conjectures, entre autres que $g(\sigma)$ est monotone dans cette région et que son maximum, en σ_c , vaut g_ω .

III. Interpolation dimensionelle

Retournons maintenant aux séries non renormalisées des fonctions de Schwinger non tronquées. On les obtient en développant les exponentielles $\exp - \lambda \int_\Lambda d^\nu x \phi^4(x)$ dans la formule (1):

$$S^{(b)}_{NON\ REN}(x_1,\ldots x_N) = Z(\Lambda)^{-1}\{ \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_\Lambda \ldots \int_\Lambda d^\nu \xi_1 \ldots d^\nu \xi_n \tag{20}$$

$$\int \phi \exp - \frac{1}{2} \int_\Lambda [(\partial\phi)^2(x) + m_0^2\ \phi(x)] d^\nu x\ \phi^4(\xi_1)\ldots\phi^4(\xi_n) \prod_{j=1}^{N} \phi(x_j) \}$$

$$Z(\Lambda) = \sum_{n=}^{\infty} \frac{(-\lambda)^n}{n!} \int_\Lambda \ldots \int_\Lambda d^\nu \xi_1 \ldots d^\nu \xi_n \int \phi \tag{21}$$

$$\exp - \frac{1}{2} \int_\Lambda [(\partial\phi)^2(x) + m_0^2\phi^2(x)] dx\ \phi^4(\xi_1)\ldots\phi^4(\xi_n)$$

Effectuons les intégrales gaussiennes et simplifions en remplaçant par ξ_j ; on obtient:

$$S^{(N)}_{NON\ REN}(x_1\ldots x_N) = Z(\Lambda)^{-1}\{ \sum_{n=0} \frac{(-\lambda)^n}{n!} \int_\Lambda \ldots \int_\Lambda d^\nu 1 \ldots d^\nu n [x_1 \ldots x_N 1111 \ldots nnnn]\} \tag{22}$$

$$Z(\Lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_\Lambda \ldots \int_\Lambda d^\nu 1 \ldots d^\nu n [1111 \ldots nnnn] \ . \tag{23}$$

Ici le symbole [1...n] désigne un hafnien défini par récurrence à partir des objets [j;k] . Il vaut zéro pour n impair, et pour n pair:

$$[1,\ldots,n] = \sum_{j=2}^{n} [1;j][\hat{1},2,\ldots,\hat{j},\ldots n] \tag{24}$$

Dans notre cas le symbole [j;k] est la fonction à deux points euclidenne:

$$[j;k] = \Delta_E(m_0^2\ ,\ j-k) \ . \tag{25}$$

$$\Delta_E(m_0^2,x) = \frac{1}{(2\pi)^\nu} \int \frac{\exp ip\cdot x}{p^2+m^2} d^\nu x \qquad (26)$$

La formule de récurrence (24) montre qu'un hafnien d'ordre n se développe en (n-1)!! produits de hafniens élémentaires d'ordre 2 ((n-1)!! = (n-1)(n-1)(n-3)...3.1). Par conséquent le coefficient de λ^n au numérateur de (23) contient $K_{N,n} = (N+4n-1)!!$ termes. C'est à chacun de ces termes qu'est associé un diagramme de Feynman. Nous entendons par là un ensemble de n + N vertex indiciés, $V_1,\ldots,V_{n+N}$, et de lignes joignant ces vertex. Aux vertex $V_1\ldots V_{n+N}$ sont associées les variables $1,\ldots,n,x_1,\ldots,x_N$, que nous appelerons aussi respectivement $\eta_1, \eta_2,\ldots\eta_{n+N}$. A un terme du développement de $[x_1,\ldots,x_N,1111\ldots nnnn]$, on associe le diagramme avec une ligne joignant V_i à V_j pour chaque hafnien élémentaire $[\eta_i;\eta_j]$ apparaissant dans le produit en question. Les vertex $V_1,\ldots V_n$ et les lignes $[\eta_i;\eta_j]_{i\leq n \text{ et } j\leq n}$ d'un diagramme G seront dits internes et forment un diagramme noté G_{int}. Les autres seront dits vertex externes ou lignes externes, "jambes". Dans notre cas (modèle ϕ^4) un vertex interne appartient à quatre lignes, un vertex externe, à une seule. Nous appelerons les diagrammes ayant cette propriété simplement "diagrammes de ϕ^4".

On a les notions topologiques habituelles pour la figure formée par les lignes du diagramme, comme la connexité. Une notion utile plus loin est celle de diagramme 2-connexe: c'est un diagramme connexe qu'on ne peut séparer en deux parties connexes en enlevant une ligne, ou bien un vertex.

Introduisons aussi le nombre de boucles indépendantes ou 1^e nombre de Betti du diagramme, qui s'écrit:

$$b(G) = \ell(G) - n(G) + c(G) = b(G_{int}) \qquad (27)$$

où $\ell(G)$, $n(G)$, $c(G)$ désignent le nombre de lignes internes, de vertex internes, et de composantes connexes contenant des vertex internes, de G.

On a d'ailleurs aussi:

$$b(G) = (\ell(G)+L(G)) - (n(G)+N(G)) + C(G) \tag{28}$$

$L(G)$, $N(G)$, $C(G)$ désignant le nombre de lignes externes, de vertex externes, et de composantes connexes de G. Avec ces notions notre développement prend la forme:

$$S^{(N)}_{\text{NON REN}}(x_1 \ldots x_N) = Z(\Lambda)^{-1}\{\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \sum_{\substack{G \text{ diagramme de } \phi^4 \\ \text{tel que } n(G)=n, N(G)=N}} I_G(x_1 \ldots x_N) \tag{29}$$

$$I_G(x_1 \ldots x_N) = \int_\Lambda \ldots \int_\Lambda d^\nu \eta_1 \ldots d^\nu \eta_n \prod_{\ell \text{ ligne de } G} \Delta_E\Big(\sum_{j=1}^{n+N} [j:k]\eta_j\Big) \tag{30}$$

où $\{(j,k), j=1,\ldots,n+N; k=1,\ldots \ell+L = \frac{4n+N}{2}$ est la matrice d'incidence définie, après orientation arbitraire des lignes de G par:

$$[j:k] = \begin{cases} +1 & \text{si la ligne } k \text{ arrive au vertex } j \text{ mais n'en part pas} \\ -1 & \text{si la ligne } k \text{ part du vertex } j \text{ mais n'y arrive pas} \\ 0 & \text{dans les autres cas.} \end{cases}$$

Remarquons qu'à chaque diagramme est ainsi associée une matrice, mais celle ci ne le caractérise en général que si le diagramme ne présente pas de "tad-pôles" ; pourtant si l'on sait que le diagramme est "de ϕ^4", sa matrice d'incidence le détermine complètement. On peut vérifier que les fonctions de Schwinger tronquées introduites au paragraphe I ont un développement analogue mais où n'interviennent plus que des diagrammes connexes:

$$S^{(N)}_{\text{NON REN}}(x_1 \ldots x_N) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \sum_{\substack{G \text{ diagramme de} \\ \text{connexe tel que } n(G)-n, N(G)=N}} I_G(x_1 \ldots x_N) \qquad (31)$$

De même la quantité p (3) qui correspondrait à la pression grand-canonique dans le problème de mécanique statistique associé, mais qui dans notre contexte de théorie des champs serait appelée plus justement "densité d'action" admet pour développement:

$$p = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ell n\, Z(\Lambda) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int d^\nu 2 \ldots d^\nu n\, [11112222\ldots nnnn]^T \qquad (32)$$

où le hafnien tronqué $[1\ldots n]^T$ est défini à partir du hafnien entier par une formule analogue à (5); son développement est indicié par les diagrammes connexes sans "jambes" à n vertex. On remarquera qu'on à utilisé l'invariance de $[11112222\ldots nnnn]^T$ par translation globale pour intéger sur la variable 1; ceci explique aussi la division par $|\Lambda|$ dans (3) et (32), en vue de construire une "densité d'action" finie.

Nous allons étudier les amplitudes I_G non-renormalisées ("nues"), pour G connexe, qui apparaissent dans le développement (31). Passons en transformée de Fourier en introduisant des variables $p_1, \ldots, p_N$ duales de $x_1, \ldots x_N$, et remplaçons Δ_E par son expression (26):

$$I_G = \frac{1}{(2\pi)^{\nu(\ell+L)}} \int d^\nu \eta_1 \ldots d^\nu \eta_{n+N}\, d^\nu q_1 \ldots d^\nu q_{\ell+L} \prod_{d=1}^{\ell+L} \frac{1}{q^2 k + m^2}$$
$$\exp(-i \sum_{j=1}^{N} p_j \eta_{j+n}) \exp[i \sum_{k=1}^{\ell+L} q_k (\sum_{j=1}^{n+N} [j:k] \eta_j)] \qquad (33)$$

Les intégrations sur les variables η effectuées, on obtient:

$$I_G(p_1,\ldots,p_N) = \frac{1}{(2\pi)^{\nu(\ell+L)}} \int d^\nu q_1 \ldots d^\nu q_{\ell+L} \prod_{k=1}^{\ell+L} \left(\frac{1}{q_k^2+m^2}\right) \prod_{j=1}^{n+N} \delta\left(p_j - \sum_{k=1}^{\ell+L} [j:k] q_k\right) \qquad (34)$$

La véritable intégration se fait donc sur le sous espace défini par $(p_j - \sum_{k=1}^{\ell+L} [j:k] q_k) = 0 \quad (j=1,\ldots,n+N)$. Comme $\sum_{j=1}^{n+N} [j:k] = 0$ pour toute ligne k, ces équations entraînent

$$\sum_{j=1}^{N} p_j = 0 \qquad (35)$$

I_G est donc nul sauf s'il y a conservation du moment total c'est à dire si $\underline{p} = (p_1,\ldots p_N)$ satisfait la condition (35) ce qu'on supposera désormais. Il nous reste alors (n+N-1) fonctions δ indépendantes dans (34), car G est connexe. Elles expriment la conservation des impulsions à chaque vertex. (Notons par conséquent que $\prod_{k=1}^{\ell+L} \frac{1}{(q_k^2+m^2)}$ n'est fonction que de la matrice $(\vec{p}_i \cdot \vec{p}_j)_{i,j=1,\ldots N}$ des invariants scalaires associés à $\underline{p}$.) On peut éliminer ces fonctions δ en intégrant sur n+n-1 variables; on conserve donc $(\ell+L) - (n+N-1) = b$ variables d'intégration, une par boucle indépendante. Mais le changement de variables introduisant ces variables de boucles privilégie arbitrairement certaines lignes du diagramme. Pour garder une forme aussi symétrique que possible nous préférons utiliser la représentation paramétrique de Schwinger. On écrit le propagateur $\frac{1}{q_k^2+m^2}$ sous la forme:

$$\frac{1}{q_k^2+m^2} = \int_0^\infty d\alpha_k \exp[-\alpha_k(q_k^2+m^2)] \qquad (36)$$

si k est une ligne interne. En inversant les intégrations en α et q et en effectuant ces dernières on obtient:

$$I_G(\underline{p}) = \frac{1}{(2\pi)^{\nu(\ell+L)}} \prod_{j=1}^{N} \frac{1}{p_j^2+m^2} \int_{[0,\infty)^\ell} \prod_{k=1}^{\ell} d\alpha_k$$

$$\frac{1}{[U_G(\alpha)]^{\nu/2}} \exp\left[- \sum_{k=1}^{\ell} \alpha_k m - V_G(\underline{p},\alpha) \right] \tag{37}$$

Nous oublierons le plus souvent le facteur multiplicatif $\frac{1}{(2\pi)^{\nu(\ell+L)}} \prod_{j=1}^{N} \frac{1}{p_j^2+m^2}$ dans I_G, qui est sans grande importance pour la suite. On appelera $d\mu_m(\alpha)$ la mesure $\prod_{k=1}^{\ell} d\alpha_k \exp - \sum_{k=1}^{\ell} \alpha_k m^2$ sur $[0,\infty)^\ell$. Les fonctions U_G et V_G ont été calculées par Symanzik:

$$U_G(\alpha) = \sum_{T \text{ arbre de } G} \prod_{k \notin T} \alpha_k \tag{38}$$

$$V_G(\underline{p},\alpha) = \frac{-1}{U_G(\alpha)} \sum_{\Delta \text{ coupure de } G} \left(\prod_{k \in \Delta} \alpha_k \right) U_{G_1}(\alpha) U_{G_2}(\alpha) \vec{P}_{G_1} \cdot \vec{P}_{G_2} \tag{39}$$

Un arbre de G est un sous diagramme de G_{int} connexe sans boucle contenant tous les vertex de G_{int}. Une coupure est un ensemble de lignes de G_{int}, tel qu'après leur suppression dans G_{int} on obtienne exactement deux sous diagrammes de G_{int}, G_1 et G_2 connexes. $\vec{P}_{G_1}$ et $\vec{P}_{G_2}$ désignent la somme des impulsions externes de $\underline{p}$ arrivant aux vertex de G_1 et de G_2. A cause de la conservation du moment (35) on a d'ailleurs $\vec{P}_{G_1} = - \vec{P}_{G_2}$ ce qui montre que $V_G(\underline{p},\alpha)$ est positif, l'espace étant euclidien. Les fonctions U_G et V_G sont globalement homogènes en α, de degré respectivement $b(G)$ et 1.

Nous remarquons que la dimension d'espace-temps, ν, n'apparaît plus que comme l'exposant de $(U_G(\alpha))^{-\frac{1}{2}}$ dans la formule (37). I_G n'étant fonction que des invariants scalaires de $\underline{p}$ la dimension des moments n'est présente qu'indirectement, limitant le rang de la matrice $(\vec{p}_i \cdot \vec{p}_j)$. Ceci suggère

de considérer ν comme variable complexe dans (37): cette interpolation naturelle a été notamment étudiée par Speer [6] qui a montré que l'intégrale (37) est absolument convergente dans un demi plan $\mathrm{Re}\ \nu < \nu_0(G)$, et y définit une fonction analytique, prolongeable en fonction méromorphe dans tout le plan complexe, dont les pôles sont des réels rationnels, dont il a donné la position.

Plus précisément si G est un diagramme de ϕ^4 2-connexe, I_G présente des pôles propres aux rationnels de la forme:

$$\nu = 4 + \frac{N(G)-4+2k}{1+n(G) - \frac{N(G)}{2}} \qquad k \in \mathbb{N} \qquad (40)$$

l'ensemble des pôles de I_G pour un diagramme G quelconque est la réunion des pôles propres de tous les sous diagrammes 2-connexes de G .

En particulier voici la situation des pôles propres d'un diagramme 2-connexe G pour les premières valeurs de $N(G)$:

$$N(G) = 0 \qquad \nu = 4 - \frac{4}{n+1} + \frac{2k}{n+1} \qquad k = 0,1,2...$$

$$N(B) = 2 \qquad \nu = 4 - \frac{2}{n} + \frac{2k}{n} \qquad k = 0,1,2...$$

$$N(G) = 4 \qquad \nu = 4 + \frac{2k}{n-1} \qquad k = 0,1,2...$$

Le point $\nu = 4$ est donc toujours le premier pôle propre d'un diagramme 2-connexe de la fonction a 4 points, le deuxième pôle propre d'un diagramme 2-connexe de la fonction à 2 points et le troisième pôle propre d'un diagramme 2-connexe de la pression. Il a d'autres particularités:

- L'ensemble des pôles $\nu < 4$ est discret; celui des pôles $\nu \geq 4$ est dense dans $[4,\infty)$ (tous les rationnels ≥ 4)
- Un rationnel $\nu > 4$ ne peut être, éventuellement, qu'un

pôle propre d'un diagramme 2-connexe à zéro ou deux "jambes". Au contraire tout rationnel $\nu > 4$ est pour tout N un pôle propre d'au moins un diagramme à N vertex externes.

Indiquons les pôles des premières amplitudes pour la fonction à 2 points:

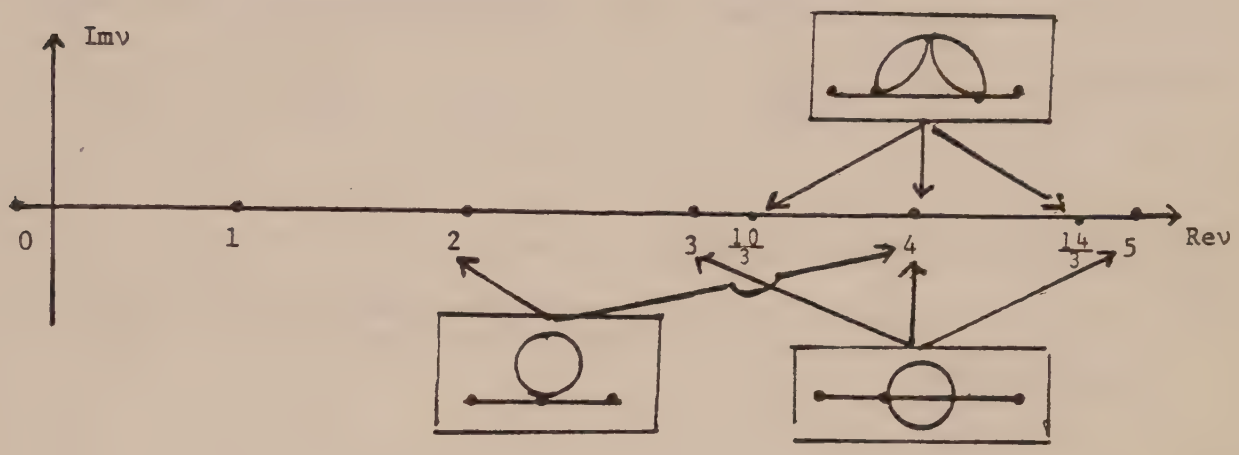


Figure 3 Pôles des diagrammes à 2 lignes externes

Si l'on ajoutait de plus en plus de graphes on verrait nettement 4 apparaître comme premier point d'accumulation.

Lorsque $\mathrm{Re}\ \nu$ est supérieur ou égal au premier pôle, $\nu_0(G)$, l'intégrale (37) n'est plus absolument convergente (comportement de $\frac{1}{U_G^{\nu/2}}$ lorsque certains α approchent 0). Il faut effectuer une renormalisation. Ignorons encore provisoirement ce problème et, pour explorer l'idée d'interpolation en dimensions, acceptons l'intégrale (37) comme définissant pour ν réel une fonction positive des invariants de $\underline{p}$, valant éventuellement $+\infty$. On voit que:

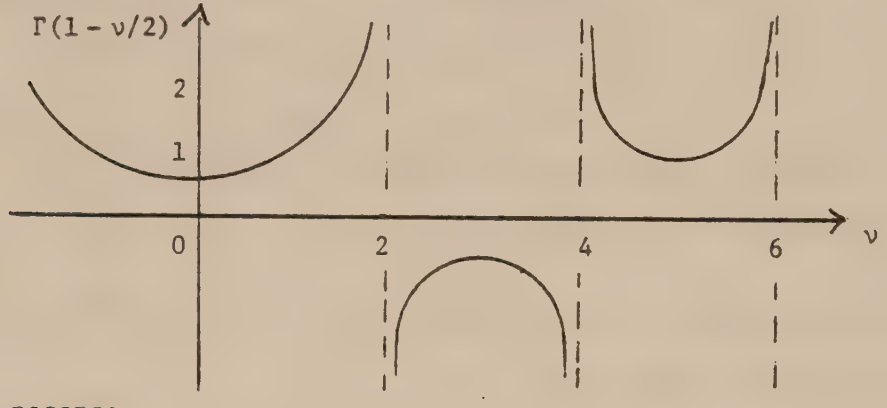
$$\frac{\partial^2}{\partial \nu^2} I_G = \int du_m(\alpha) [\ln U_G(\alpha)]^2 \frac{\exp -V_G(\underline{p},\alpha)}{[U_G(\alpha)]^{\nu/2}} \geq 0 \qquad (41)$$

Ainsi I_G est convexe en ν . On pourrait donc penser estimer par exemple $I_G(\nu)$ en termes de $I_G(1)$ et $I_G(3)$ pour $1 \leq \nu \leq 3$. En fa: cette remarque s'avère inutile parce qu'en général la renormalisation, qui nous sera nécessaire, ne respecte pas la positivité de l'intégrand (41.)

Pour mieux comprendre cet effet nous étudions les amplitudes des graphes et (diagrammes dont on oublie l'indication et les lignes externes).

$$\text{ } = \lambda\mu^{\mu-\nu} \frac{m_0^{\nu-\mu}}{(4\pi)^{\nu/2}} \Gamma(1-\nu/2)$$

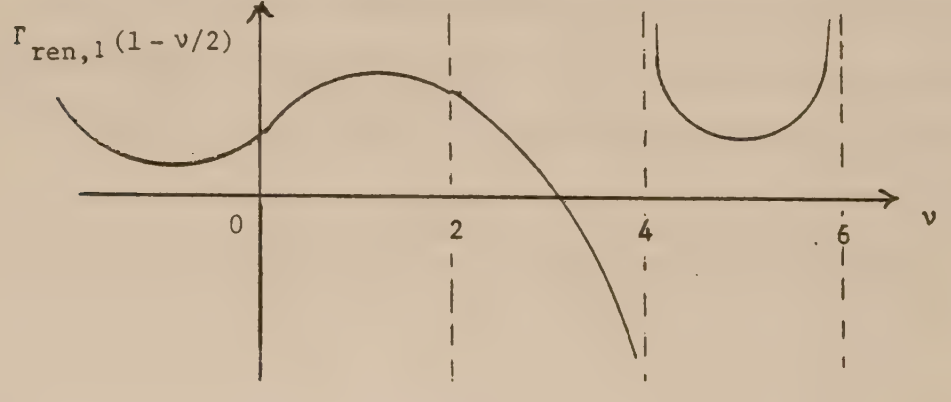
(μ est un paramètre positif). La fonction Γ a des pôles à $(1 - \nu/2) = 0,-1,-2$, c'est à dire à $\nu = 2,4,6...$, de résidus -1, 1/3, - 1/3!, ... respectivement:



Si nous posons:

$$\Gamma_{ren,1}\ (1 - \nu/2) = \Gamma(1 - \nu/2) + \frac{1}{\nu/2 - 1} \qquad (42)$$

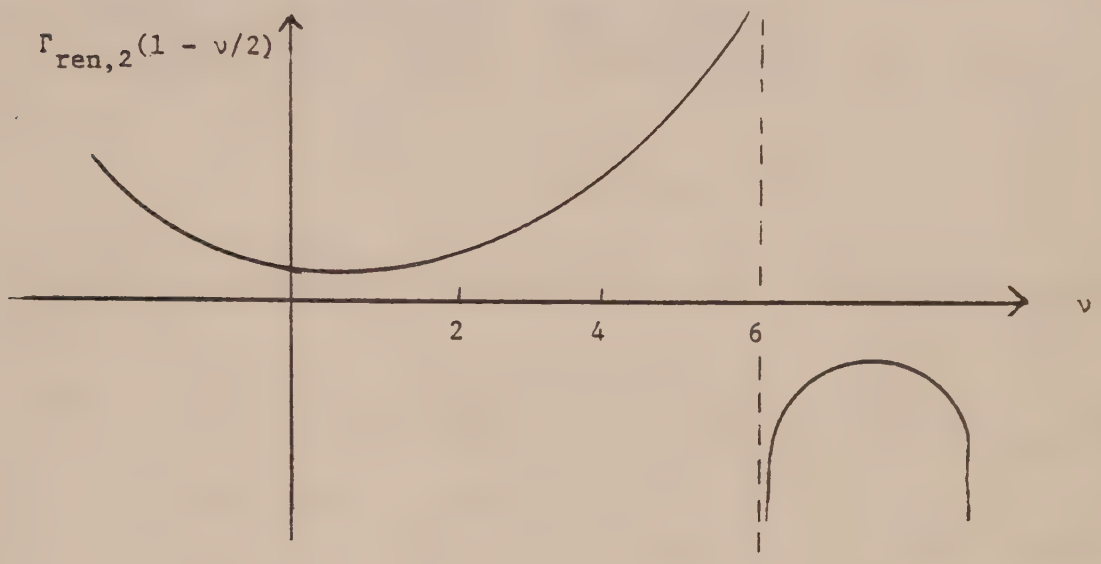
nous obtenons une fonction dont le comportement est:



Si nous posons

$$\Gamma_{ren,2}(1-\nu/2) = \Gamma(1-\nu/2) + \frac{1}{\nu/2-1} - \frac{1}{2(\nu/2-2)} \qquad (43)$$

nous obtenons une fonction convexe et positive dans l'intervalle $\nu < 6$:



Bergère et David ont suggéré une représentation intégrale de la fonction:

$$\Gamma_{ren,N}(x) = \Gamma(x) - \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \frac{1}{x+n}$$

où $$\Gamma_{ren,0}(x) = \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (44)$$

à savoir $$\Gamma_{ren,N}(x) = \int_1^\infty t^{x-1} e^{-t} dt + \int_0^1 t^{x-1} \left[e^{-t} - \sum_{n=0}^{N-1} \frac{(-t)^n}{n!} \right] dt$$

Il est facile de montrer que pour N pair, $\Gamma_{ren,N}$ est convexe et positive sur l'intervalle $-N < x < +\infty$. Par conséquent en posant

$$[\Omega]_{ren,N} = g\, \mu^{4-\nu} \frac{m_0^{\nu-2}}{(4\pi)^{\nu/2}} \Gamma_{ren,N}(1-\nu/2)$$

alors $[\bigcirc]_{ren,1}$ est une amplitude bien définie dans $-\infty < \nu < 4$, mais non convexe ; $[\bigcirc]_{ren,2}$ est bien définie dans $-\infty < \nu < 6$ et reste convexe, mais nous allons voir qu'une difficulté importante apparaît, dûe à la soustraction "supplémentaire" effectuée, lorsqu'on essaie de généraliser ceci. Ainsi considérons le graphe, $\ominus$, dont l'amplitude, a une constante près, est:

$$\ominus = \int_0^\infty\int_0^\infty\int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(U(\alpha))^{\nu/2}} \exp\left[-\frac{\alpha_1\alpha_2\alpha_3 p^2}{U(\alpha)} - m^2(\alpha_1+\alpha_2+\alpha_3)\right]$$

où $U(\alpha) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$

Les analogues de (42) et (43) sont, en notant θ fonction de Heaviside:

$$[\ominus]_{ren,1}(p) = \int_0^\infty\int_0^\infty\int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(U(\alpha))^{\nu/2}} \left\{\exp\left[-\frac{\alpha_1\alpha_2\alpha_3 p^2}{U(\alpha)} - m^2(\alpha_1+\alpha_2+\alpha_3)\right] - \theta(1-U(\alpha))\right\} \qquad (45)$$

convergent pour $-\infty < \nu < 4$, et :

$$[\ominus]_{ren,2}(p) = \int_0^\infty\int_0^\infty\int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(U(\alpha))^{\nu/2}} \left\{\exp\left[-\frac{\alpha_1\alpha_2\alpha_3 p^2}{U(\alpha)} - m^2(\alpha_1+\alpha_2+\alpha_3)\right] - \theta(1-U(\alpha))\left[1 - \frac{\alpha_1\alpha_2\alpha_3 p^2}{U(\alpha)} - m^2(\alpha_1+\alpha_2+\alpha_3)\right]\right\} \qquad (46)$$

convergent pour $-\infty < \nu < 5$. $[\bigcirc]_{ren,2}$ donne, comme $[\bigcirc]_{ren,2}$ une fonction positive convexe en ν dans son domaine de définition. Mais il apparaît un terme proportionnel à p^2 dans (46) [7]. Introduit dans un diagramme plus compliqué, $[\bigcirc]_{ren,2}$ exigerait un procédé de renormalisation du champs aussi bien que de renormalisation de la masse, qui est seul nécessaire lorsqu'on utilise $[\bigcirc]_{ren,1}$. Comme la simplicité de notre méthode plus loin repose sur le fait que seules des renormalisations de

masse, en nombre fini sont nécessaires, nous abandonnons $[\ominus]_{ren,2}$. Systématisons maintenant la procédure de renormalisation que nous allons utiliser dans la region $Re\ \nu < 4$. Bergère, Lam et Zuber [8] ont exprimé le procédé habituel, satisfaisant la récurrence BPHZ, à l'aide d'un opérateur R agissant directement sur l'intégrand de (37). Une forme particulièrement compacte pour cet opérateur est:

$$R = \prod_{S \text{ sous ensemble de } [1,\ldots,\ell(G)]} (1-\tau_s^{-\ell(s)}) \qquad (47)$$

où l'opérateur de Taylor généralisé pour les variables de S , τ_s , est défini par:

$$\tau_s^n f(\alpha) = [\tau_\rho^n f(\alpha,\rho)]_{\rho=1}$$

où $f(\rho,\alpha)$ est la fonction obtenue en dilatant les α relatifs aux lignes de S : $\alpha_k \to \rho\alpha_k$ pour $k \in S$. Si $f(x)$ est une fonction de x réél, telle que $x^\nu f(x)$ (ν complexe) soit C^∞ en $x = 0$, $\tau_x^n f$, ($n \in Z$) est la fonction:

$$[\tau_x^n f](x) = x^{-\lambda-\varepsilon}\ T_x^{n+\lambda}\ \{x^{\lambda+\varepsilon} f(x)\}$$

quel que soit λ entier , $\lambda \geq E(Re\ \nu)$; $\varepsilon = \nu - E(Re\ \nu)$, E désignant la partie entière, et T_x^n l'opérateur de Taylor usuel en $x = 0$: $[T_x^n f](x) = \sum_{i=0}^{n} \frac{x^n}{n!} f^{(i)}(0)$.

Diverses formes équivalentes de l'opération R sont discutées dans [8]; remarquons que (47) ne fait intervenir la topologie de G que par l'intermédiaire du nombre de ses lignes internes.

L'amplitude renormalisée selon le schéma ordinaire de soustraction dit "à impulsions externes nulles" est alors, lorsque ν est entier:

$$I_G^R(\underline{p}) = \int d\mu_m(\alpha)\ \mathbf{R}\left[\frac{\exp - V_G(\underline{p},\alpha)}{(U_G(\alpha))^{\nu/2}}\right] \qquad (48)$$

Bergère et David ont étudié la généralisation de cette formule à ν complexe. Pour un diagramme G donné, les pôles sont des rationnels isolés que l'on peut ordonner, $\nu_0,\nu_1,\ldots\nu_i\ldots$ $i \in \mathbb{N}$. L'intégrale (48) est absolument convergente pour $\nu \in \mathbb{C}$ et définit une fonction analytique de ν dans chaque bande $\nu_i < \mathrm{Re}\ \nu < \nu_{i+1}$. Cette fonction désignée par $I_G^{\nu_i}$ admet un prolongement analytique dans le demi plan $\mathrm{Re}\ \nu < \nu_{i+1}$. En général si $\mathrm{Re}\ \nu < \nu_{i+1}$ $I_G^{\nu_i}(\nu)$ et $I_G^{\nu_{i+k}}(\nu)$, $k \in \mathbb{N}^*$ différent, car $I_G^{\nu_{i+k}}$ contient davantage de soustractions. Plus généralement pour Δ réél on peut considérer

$$I_G^{\Delta}(\underline{p},\nu) = I_G^{\nu_i(\Delta,G)}(\underline{p},\nu) \quad \text{où} \quad \nu_i(\Delta,G)$$

est le plus grand pôle de G inférieur ou égal à Δ. C'est une fonction analytique dans un demi plan ouvert contrenant Δ, à savoir $\mathrm{Re}\ \nu < \nu_{i+1}(\Delta,G)$, premier pôle de G après $\nu_i(\Delta,G)$. Pour le modèle ϕ^4 et lorsque $\Delta < 4$, l'ensemble $\mathcal{P}$ des pôles strictement inférieurs à 4, discret, contient un plus petit élément strictement supérieur à Δ, $\nu_s(\Delta)$. Alors I_G^{Δ} est analytique dans un demi plan ouvert contenant Δ indépendant de G, à savoir $\mathrm{Re}\ \nu < \nu_s(\Delta)$. Cette dernière propriété est intéréssante puis que nous avons en vue de sommer les amplitudes de diagrammes différents afin de dépasser le cadre perturbatif.

Pour tout $\varepsilon > 0$, $I_G^{4-\varepsilon}(\underline{p},\nu)$ est donc une interpolation dimensionelle raisonnable dans la région $\mathrm{Re}\ \nu < \nu_s(4-\varepsilon)$ pour l'amplitude renormalisée associée à G, et fournit une première manière de bien définir notre problème: la série des perturbations de ϕ^4 est elle Borel sommable pour toutes les

dimensions ou le modele est superrenormalisable? Il suffira en effet d'examiner si pour tout $\varepsilon > 0$ la série (31) où les amplitudes sont maintenant $I_G^{4-\varepsilon}(\underline{p},\nu)$ est Borel sommable pour $\mathrm{Re}\,\nu < \nu_s(4-\varepsilon)$ (ou au moins pour $\mathrm{Re}\,\nu = 4-\varepsilon$ car nous allons voir que les deux problèmes sont équivalents).

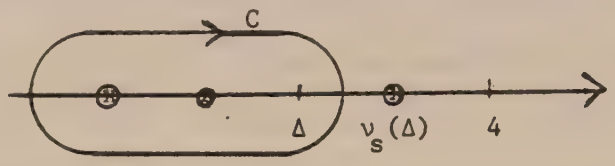
Néanmoins une interpolation dimensionelle plus naturelle encore est celle où l'on renormalise dimensionellement, c'est à dire où les contres termes satisfont d'une part au schéma récursif BPHZ , et d'autre part sont exactement des parties divergentes des séries de Laurent exprimant les amplitudes nues au voisinage des différents pôles. Cette procédure est définie et étudiée en détail dans [6]. Un de ses grands avantages est de permettre de traiter les théories plus complexes et physiquement plus réalistes que ϕ^4 , possédant des symétries de gauge, sans que la renormalisation détruise ces symétries.

Montrons comment nous voulons utiliser ce schéma de renormalisation. Selon [6] , la partie singulière du développement de Laurent d'une fonction méromorphe près d'un pôle est donnée par l'intégrale de contour:

$$K(f) = + \frac{1}{2\pi i} \int_C \frac{f(\nu')}{\nu-\nu'} \, d\nu'$$

où C est un contour entourant uniquement le pôle en question. Dans [6] ce pôle est $\nu = 4$ car c'est les théories habituelles en dimension 4 qui sont en vue. On définit alors des contre termes comme partier singulières des sous graphes divergents d'un graphe, et on les soustrait dans l'amplitude I_G de façon à obtenir un résultat fini et à satisfaire aux conditions BPHZ. (Notons que cette dernière condition ne serait pas satisfaite si l'on se contenait de prendre pour amplitude renormalisée $I_G - K(I_G)$) .

Nous definissons par contre des amplitudes $I^{\Delta}_{G\ DIM}(\underline{p},\nu)$ analytiques pour $Re\ \nu < \nu_s(\Delta)$ où nous extrayons tous les pôles situés avant $\nu_s(\Delta)$ par cette procédure et non plus seulement pôle $\nu = 4$. $I^{\Delta}_{G\ DIM}(\underline{p},\nu)$ est donc défini exactement de la manière récursive exposée dans [6] mais où les parties singulières retranchées s'expriment à l'aide d'un contour englobant tous les pôles inférieurs à $\nu_s(\Delta)$:



Les arguments de [6] s'appliquent alors, et prouvent que nos $I^{\Delta}_{G\ DIM}(\underline{p},\nu)$ satisfont bien la récurrence BPHZ et sont analytiques pour $Re\ \nu < \nu_s(\Delta)$.

Bien que l'on ne dispose pas encore d'une forme compacte du type (47) exprimant la renormalisation dimensionelle sous la forme d'un opérateur agissant sur l'intégrand en α, cette question est étudiée activement (Pour $\nu = 4$) [9]. La solution devrait s'adapter facilement à notre problème et nous permettre d'écrire de façon compacte on représentation α les amplitudes $I^{\Delta}_{G\ DIM}(\underline{p},\nu)$. Les formules (42) et (45) sont en fait un premier exemple où l'on a trouvé une telle forme pour les amplitudes des graphes simples et .

Notre problème peut se traiter indifféremment en utilisant les séries (31) construites avec les amplitudes $I^{R}_{G}(\nu)$, $I^{\Delta}_{G}(\nu)$ ou $I^{\Delta}_{G\ DIM}(\nu)$, pour tout Δ tel que $\nu \leqslant \Delta < 4$. (Notons d'ailleurs que $I^{R}_{G}(\nu) = I^{\Delta}_{G}(\nu)$ si $\nu_i(G,\nu) \leqslant \Delta < \nu_s(\nu)$. En effet pour une dimension ν fixée, $Re\ \nu < 4$, les problèmes de sommation de Borel de ces différentes séries sont tous équivalents entre eux. Ceci tient à ce que les prescriptions de renormalisation pour I^{Δ}_{G} et $I^{\Delta}_{G\ DIM}$ satisfont le schéma récursif

BPHZ [7] [8] ; nous savons qu'elles doivent se traduire par une simple différence finie dans les masses renormalisées des différentes théories obtenues; cette différence finie s'exprime elle même comme un polynôme dans la constante de couplage nue, λ , puisque l'on reste dans le cadre de théories superrenormalisables. Un tel polynôme est évidemment Borel sommable, et certaines résultats généraux [10] assurent que la composée de deux fonctions Borel sommables est Borel sommable.

Au prochain paragraphe nous exposons un premier résultat conduisant à l'analyticité de la transformée de Borel de la série perturbative près de l'origine, mais pour l'instant seulement dans la région $\mathrm{Re}\ \nu < \frac{10}{3}$. Dans ce domaine limité et pour $\Delta < \frac{10}{3}$, les prescriptions (42) et (45) sont suffisantes pour définir $I^{\Delta}_{G\ DIM}(\underline{p},\nu)$ et il ne sera pas difficile de vérifier que notre résultat s'applique effectivement aussi bien à $I^{\Delta}_{G}(\nu)$ qu'à $I^{\Delta}_{G\ DIM}(\nu)$.

A titre d'exemple comparons, dans ce domaine, les différentes amplitudes que nous avons introduites pour le graphe = G :

$\Delta < 3$

$$I^{\Delta}_{G}(\underline{p},\nu) = I^{\Delta}_{G\ DIM}(p,\nu) = \int d\mu_m(\alpha_1,\alpha_2,\alpha_3)\ \frac{1}{[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3]^{\nu/2}} \exp\left(-\frac{p^2\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}\right) \tag{49}$$

analytiques pour $\mathrm{Re}\ \nu < 3$.

$3 \leq \Delta < \frac{10}{3}$

$$I_G^{\Delta}(p,\nu) = \int d\mu_m(\alpha_1,\alpha_2,\alpha_3) \frac{1}{[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3]^{\nu/2}} \left\{ \exp\left[- \frac{p^2\alpha_I\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3} \right] - 1 \right\} \tag{50}$$

$$I_{G\ DIM}^{\Delta}(p,\nu) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3]^{\nu/2}}$$

$$\left\{ \exp\left[- \frac{p^2\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3) \right] - \theta\left[1 - (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\right] \right\} \tag{51}$$

analytiques pour $\mathrm{Re}\ \nu < \frac{10}{3}$.

IV. Estimations d'amplitudes

Nous prendrons pour point de départ la forme (48) d'une amplitude de la théorie ϕ^4 ; nous montrerons une borne uniforme, exponentielle dans le nombre de vertex de G , ceci pour l'instant dans le domaine $\mathrm{Re}\ \nu < \frac{10}{3}$. Nous verrons pourquoi cette majoration est reliée très simplement au problème de l'existence de la transformée de Borel de la série perturbative dans un disque centré a l'origine. Bien que ces résultats ne constituent pas une surprise, de telles majorations n'ont semble t'il jamais été établies, excepté en dimension 1 par Bender, Wu [11] et Simon [12]. La référence [13] fournit néanmoins des résultats voisins, dans le cadre de la théorie minkovskienne et par une méthode différente.

Nous énonçons ces majorations dans l'espace des moments:

Proposition Il existe un nombre A_ν positif tel que si n désigne le nombre de vertex d'un graphe G de la théorie ϕ^4 , on ait:

$$|I_G^R(p,\nu)| \leq A_\nu^n , \qquad \forall p , \ \forall \nu / \mathrm{Re}\ \nu < \frac{10}{3} \tag{52}$$

où $I_G^R(p,\nu)$ est défini par la formule (48) . Nous montrerons d'abord le résultat pour $\mathrm{Re}\ \nu < \frac{8}{3}$.

Pour $\mathrm{Re}\ \nu < 2$ aucune renormalisation n'est effectuée. Les arguments de Bender et Wu s'étendent sans aucune modification. Pour $2 \leq \mathrm{Re}\ \nu < \frac{8}{3}$, la renormalisation est équivalente à l'ordre de Wick et se contente d'annuler les contributions des graphes comprenant des "tad pôles" . Notons que $\frac{8}{3}$ correspond au premier graphe divergent sans "tad pôles," à savoir . Nous aurons seulement besoin d'un lemme topologique très simple.

Lemme 1 Pour un graphe G connexe sans "tad pôles", il existe un entier k, compris entre 1 et 4, et k arbres de G différents tels que toute ligne interne de G appartienne à l'un au moins d'entre eux.

Preuve Le lemme est vrai parce que d'un vertex partent au plus 4 lignes. Montrons, par récurrence, qu'il existe une partition des lignes internes de G en au plus 4 sous ensembles $T_1, T'_2 \ldots T'_k$ $1 \leqslant k \leqslant 4$, tels que

a) $T_1, T'_2, \ldots T'_k$ sont sans boucles; T_1 est un arbre.

b) T'_j, $2 \leqslant j \leqslant k$ peut être complété par des éléments de T_1 de façon à former un arbre T_j ; donc $T'_j \subset T_j \subset T'_j \cup T_1$.

Supposons en effet $T_1, T'_2 \ldots T'_j$ construits vérifiant a et b (on commence par prendre n'importe quel arbre pour T_1); supposons de plus que $G - T_1 \cup T'_2 \ldots \cup T'_j$ compte au plus $4-j$ lignes par vertex (comme T_1 est un arbre, $G - T_1$ compte au plus 3 lignes par vertex). Nous construisons T'_{k+1} comme l'union d'arbres quelconques pris dans chaque composante connexe de $G - T_1 \cup T'_2 \ldots \cup T'_j$, s'il en existe de non vides. Pour compléter T'_{j+1} en un arbre T_{j+1} de G numérotons les lignes $\ell_1, \ldots \ell_r$ de T'_{k+1} ; ℓ_i point les vertex V_i et V'_i , $1 \leqslant i \leqslant r$. Il existe un chemin unique dans T_1 joignant V_1 à V'_1 ; supprimons au hasard une ligne de ce chemin; nous obtenons un graphe T^1_{j+1} , et $T^1_{j+1} \cup \{\ell_1\}$ est un arbre. Ayant ainsi construit T^s_{j+1} tel que $T^s_{j+1} \subset T_1$ et $T^s_{j+1} \cup \{\ell_1, \ell_2, \ldots \ell_s\}$ soit un arbre, il y a un chemin unique dans $T^s_{j+1} \cup \{\ell_1, \ell_2, \ldots \ell_s\}$ joignant V_{s+1} à V'_{s+1} . Ce chemin n'est pas formé exclusivement de lignes de $\{\ell_1, \ldots, \ell_s\}$ car T'_{j+1} comporterait une boucle. On peut donc supprimer une ligne de ce chemin appartenant à T_1 , et construire T^{s+1}_{j+1} , donc finalement T^r_{j+1} , et $T_{j+1} = T'_{j+1} \cup T^r_{j+1}$.

La récurrence s'achève pour $k \leqslant 4$ car on achève de dépouiller chaque vertex de toutes ses lignes en au plus 4 opérations. Il faut bien sûr que G ne contienne pas de "tad pôles".

Revenant à la représentation de Schwinger (48) de $I_G^R(p,\nu)$, il nous faut borner, pour $2 \leqslant \mathrm{Re}\ \nu < \frac{8}{3}$:

$$I_G^R(p,\nu) = \int d\mu_m(\alpha)[\exp - V_G(p,\alpha)] \frac{1}{U_G(\alpha)^{\nu/2}}$$

Comme $\exp - V_G(p,\alpha) \leqslant 1$, et en utilisant le changement de variables $\alpha_i \to \frac{\alpha_i}{m^2}$:

$$\left|I_G^R(p,\nu)\right| \leqslant [m^2]^{N/2-n-1} \int d\mu_1(\alpha) \frac{1}{\left[\sum_{\text{arbres } T} \left(\alpha \prod_{i \notin T} \alpha_i\right)\right]^{\frac{\mathrm{Re}\,\nu}{2}}}$$

$$\leqslant [m^2]^{N/2-n-1} \int d\mu_1(\alpha) \frac{1}{\left[\sum_{j=1}^{k} \left(\prod_{i \notin T_j} \alpha_i\right)\right]^{\frac{\mathrm{Re}\,\nu}{2}}}$$

$$\leqslant [m^2]^{N/2-n-1} \frac{1}{k^{\mathrm{Re}\,\nu/2}} \int d\mu_1(\alpha) \left[\prod_{j=1}^{k} \prod_{\alpha_i \notin T_j} (\alpha_i)^{\frac{-\mathrm{Re}\,\nu}{2k}}\right]$$

par une inégalité de convexité: $A_1+\ldots+A_k \leqslant k\, A_1^{1/k} \ldots A_k^{1/k}$. Donc

$$\left|I_G^R(p,\nu)\right| \leqslant [m^2]^{N/2-n-1} \left[\sup_{t \in [0,3/8\ \mathrm{Re}\,\nu]} \int_0^\infty \frac{\exp-\alpha}{\alpha^t}\, d\alpha\right]^{\ell(G)}$$

puisque $1 \leqslant k \leqslant 4$.

Comme $\ell(G) = 2n(G) - \frac{N(G)}{2}$:

$$\left|I_G^R(p,\nu)\right| \leqslant [m^2]^{N/2-n-1} \left[\Gamma\left(1 - \frac{3\mathrm{Re}\,\nu}{8}\right)\right]^{2n - N/2}$$

majoration assez fine (comparée au comportement asymptotique de I_G^R lorsqu'il est connu) et dont on déduit aussitôt le résultat cherché.

Pour établir notre proposition dans le bande $\frac{8}{3} \leqslant \text{Re } \nu < \frac{10}{3}$ par une méthode semblable, on a besoin d'un lemme topologique plus précis, et de contrôler le comportement du graphe renormalisé lorsque $\text{Re } \nu \geqslant 3$. Nous avons montré le lemme topologique suivant:

Lemme 2

Soit G un graphe différent de , , et sans sous graphes ou , avec au plus 4 lignes par vertex. Alors on a l'une des deux possibilités:

(i) G contient au plus 2 arbres tels que toute ligne interne appartient à au moins un d'éntre eux.

(ii) Il existe dans G entre 3 et 5 arbres différents tels que toute ligne interne de G appartient à au moins 2 d'entre eux.

La démonstration repose sur une discussion élémentaire, comme pour le lemme 1, mais nettement plus longue aussi nous ne la donnons pas ici mais elle est exposée en détail dans [14]. Il est facile de se convaincre que ce lemme fournit le majoration cherchée dans la bande $\frac{8}{3} \leqslant \text{Re } \nu < \frac{10}{3}$ pour les graphes G considérés dans le lemme. En effet ils n'ont pas à être renormalisés; et dans le cas (ii) par exemple, avec $T_1 \ldots T_k$ $3 \leqslant k \leqslant 5$ abres du lemme:

$$[U_G(\alpha)]^{\frac{\text{Re } \nu}{2}} \geqslant \left[\sum_{j=1}^{k} \prod_{\alpha_i \notin T_j} \alpha_i\right]^{\frac{\text{Re } \nu}{2}} \geqslant k^{\frac{\text{Re } \nu}{2}} \left[\prod_{j=1}^{k} \prod_{\alpha_i \notin T_j} \alpha_i\right]^{\frac{\text{Re } \nu}{2k}}$$

donc:

$$[U_G(\alpha)]^{-\frac{\text{Re}}{2}} \leqslant [3]^{-8/3} \prod_{i=1}^{\ell} (\alpha_i)^{-t_i} \ , \ t_i \leqslant \frac{3}{5} \ \text{Re } \frac{\nu}{2}$$

donc on aura l'analogue de la majoration (52) avec une fonction $\Gamma(1 - \frac{3 \text{ Re } \nu}{10})$ finie si Re $\nu < \frac{10}{3}$.

Il ne reste plus qu'à réduire le cas d'un graphe G avec des sous graphes $\ominus$ au cas d'un graphe qui n'en contient plus.

Remarquon tout d'abord que, si Re $\nu < 3$:

$$|I^R_{\ominus}(\vec{p})| \leq \int d\mu_m(\alpha) \frac{1}{[\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3]^{\frac{\text{Re } \nu}{2}}} \leq B$$

B étant indépendant de $\vec{p}$, moment entrant et sortant ($\ominus$ a deux "jambes" que nous ne dessinons pas). Par conséquent, si I_G a q sous graphes distincts $\ominus$ (nous excluons $\ominus$ qui est annulé si $\frac{8}{3} \leq$ Re ν) on a $|I_G| \leq B^q |I_{\tilde{G}}|$ où $\tilde{G}$ est le graphe obtenu en réduisant chaque sous graphe $\ominus$ à un vertex; donc $n(\tilde{G}) = n(G) - q$ et $\tilde{G}$ satisfait les hypothèses du lemme 2. La proposition s'en déduit aussitôt, (en exigeant que A soit supérieur à B).

Lorsque $3 \leq$ Re $\nu < \frac{10}{3}$ nous allons utiliser une borne très large qui nous suffira et se généraliserait à d'autres exemples sans difficulté. En appliquant deux fois l'inégalité de convexité:

$$\frac{1}{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3} \leq \frac{1}{3} \frac{1}{[\alpha_1\alpha_2\alpha_3]^{2/3}}$$

on obtient sans difficulté

$$|I^R_{\ominus}(\vec{p})| \leq (\tfrac{1}{3})^{\frac{\text{Re } \nu}{2}} \int_0^\infty d\mu_m \frac{1}{[\alpha_1\alpha_2\alpha_3]^{\frac{\text{Re } \nu}{3}}} (1 - \exp - p^2(\alpha_1\alpha_2\alpha_3)^{1/3}$$

Donc:

$$|I^R_{\ominus}(\vec{p})| \leq (p^2)^{(\text{Re } \nu)-3} \int_0^\infty d\mu_{m/p^2} \frac{(1-\exp-(\alpha_1\alpha_2\alpha_3)^{1/3})}{(\alpha_1\alpha_2\alpha_3)^{\frac{\text{Re } \nu}{3}}}$$

Si $3 < \mathrm{Re}\,\nu < \frac{10}{3}$, on peut majorer en remplaçant simplement $d\mu_{m/p^2}$ par $d\alpha_1 d\alpha_2 d\alpha_3$, et en utilisant la croissance de $(1-\exp -t)$, borné par t $t > 0$)

$$|I_{\ominus}^{R}(\vec{p})| \leq (p^2)^{\mathrm{Re}\,\nu - 3} \int_0^1\int_0^1\int_0^1 \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1\alpha_2\alpha_3)^{\frac{\mathrm{Re}\,\nu}{3} - \frac{1}{3}}}$$

$$+ 3 \int_0^1 d\alpha_1 \int_1^\infty d\alpha_2 \int_1^\infty d\alpha_3 \frac{1}{\alpha_1^{\frac{\mathrm{Re}\,\nu}{3} - \frac{1}{3}}} \times \frac{1}{(\alpha_2\alpha_3)^{\frac{\mathrm{Re}\,\nu}{3}}}$$

$$+ 3 \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_1^\infty d\alpha_3 \frac{1}{(\alpha_1\alpha_2)^{\frac{\mathrm{Re}\,\nu}{3} - \frac{1}{3}}} \frac{1}{\alpha_3^{\frac{\mathrm{Re}\,\nu}{3}}}$$

$$+ \int_1^\infty\int_1^\infty\int_1^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1\alpha_2\alpha_3)^{\frac{\mathrm{Re}\,\nu}{3}}} ,$$

donc:

$$|I_{\ominus}^{R}(\vec{p})| \leq C_\nu (p^2)^{\mathrm{Re}\,\nu - 3}$$

C_ν étant une fonction positive pour $3 < \mathrm{Re}\,\nu < \frac{10}{3}$. Pour $\nu = 3$ il ne serait pas difficile de montrer une borne logarithmique, $D\ \mathrm{Log}(p^2)$.

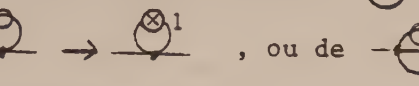
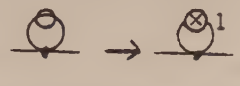
On peut regrouper ces deux cas dans une borne très large:

$$|I_{\ominus}^{R}(p^2)| \leq D_\nu + E_\nu (p^2)^{2\,\mathrm{Re}\,\nu - \frac{19}{3}}$$

où D_ν et E_ν sont des constantes positives, et $3 \leq \mathrm{Re}\,\nu < \frac{10}{3}$.

Maintenons contractons tout les sous graphes $\ominus$; comme nous excluons $\ominus$, ils apparaissent dans des chaînes $\ominus\!\!-\!\!\ominus$, que nous remplaçons par une ligne $\overset{k}{\otimes}$ où la croix centrale et l'index k rappellent qu'une chaîne de longueur k a été contractée. Le propagateur pour cette ligne est dominé par:

$$S_k(\vec{p}) = \frac{F_\nu^{\,k}}{(p^2+m^2)^{1+k(22/3 - 2\,\mathrm{Re}\,\nu)}} .$$

où F_ν est positif. Mais dans le graphe obtenu $\tilde{G}$ peuvent apparaître des tad pôles ou de nouveaux graphes  , dûs à la contraction, par exemple, de  , ou de . Cependant ces tad pôles ou graphes comportent au moins une ligne à propagateur S_k, borné par $F_\nu^k \dfrac{1}{(p^2+m^2)^{25/3 - 2 \operatorname{Re}\nu}}$.

Il n'est pas difficile de vérifier que ceci permet de contracter ces sous graphes en un vertex, en introduisant une nouvelle constante G_ν . Par exemple

$\leq F_\nu^k \int "d^\nu p" \dfrac{1}{(p^2+m^2)^{25/3 - 2 \operatorname{Re}\nu}}$

converge et est borné par $F_\nu^k\, G_\nu$ exactement à condition que $\operatorname{Re}\nu < \frac{10}{3}$.

On aplique enfin le lemme 2 à $\tilde{G}'$ obtenu après cette dernière opération, en imposant la condition $A_\nu \geq F_\nu$ et G_ν , en remarquant que $n(\tilde{G}') \leq n(G)$ - {nombre total de contractions} .

Montrons pourquoi la proposition est reliée à l'existence locale de la transformée de Borel de la série perturbative. Comme le nombre total de graphes convexes est certainement borné par $(4n + N - 1)!!$ qui est on l'a vu, le nombre de graphes total, le terme d'ordre n de notre série de Borel est borné par:

$$|a_n| \leq \frac{|\lambda|^n}{(n')^2} \cdot A_\nu^n \cdot (4n+N-1)!!$$

donc la série en question converge et définit une fonction analytique localement autour de l'origine, au moins dans le disque:

$$|\lambda| < \frac{1}{16\, A} \qquad \operatorname{Re}\nu < \frac{10}{3} \quad .$$

V. Conclusion

Les resultats de la section IV indiquent que le premier pas, dans notre programme est relativement facile. Nous avons montré que les séries perturbatives renormalisées des fonctions de Schwinger et de la densité d'action ont des transformées de Borel analytiques dans un disque du plan complexe pour la constante de couplage λ , au moins pour $\mathrm{Re}\ \nu < \frac{10}{3}$. L'extension des bornes exponentielles à $\frac{10}{3} < \mathrm{Re}\ \nu < 4$ reste un problème ouvert , mais surmontable à notre avis, qui requiert sans doute quelques raffinements des méthodes élémentaires de la partie IV.

Remarquons qu'à $\nu = 4$ il est désormais bien établi que la renormalisation détruit ce genre de bornes, certaines classes de diagrammes croissant beaucoup plus vite, au moins comme une factorielle du nombre de vertex. Néanmoins les calculations asymptotiques des grand ordres de perturbation par la méthode de Lipatov suggèrent que la transformée de Borel existe aussi dans un disque. Il n'est pas exclu que ce résultat puisse être prouvé rigoureusement par des majorations simples associées cette fois à des classes de diagrammes; mais le problème est probablement nettement plus difficile que pour $\mathrm{Re}\ \nu < 4$. Une preuve rigoureuse conférerait une légitimité accrue aux calculs qui assument ce résultat, comme ceux de la Réf. [1].

Revenons à la deuxième partie de notre programme. Il est vraissemblable que la continuation analytique le long d'une bande S_σ de la fonction que nous avons obtenue dans un disque par sa série de Taylor est un probleme très difficile. De relativement faibles modifications de la série de Taylor peuvent entraîner l'apparition de singularités près de l'axe réél positif, très loin de l'origine. Nous reconnaissons qu'il n'y a pas d'arguments excluant cette possibilité; mais disons seulement que nous

savons de facon certaine que ceci ne se produit pas aux trois points Re ν = 1,2,3, et qu'un faisceau de propriétés communes semble renforcer chez nous l'idée qu'il existe une unité du domaine superrenormalisable.

Pour prouver quelque chose dans cette direction, comme dans la plupart des exemples mathématiques de continuations analytiques, on a presque certainement besoin d'informations importantes sur la structure de la fonction obtenue dans le disque, telles que équations algébriques ou différentielles. Une première idée est d'examiner rigoureusement l'equivalent de l'équation de Callan-Symanzik dans le plan de Borel, c'est à dire en transformée de Laplace inverse, par rapport à $\frac{1}{z}$.

Si l'on considère la masse considérable de matériel mathématique et de travail nécessaire pour construire des solutions exactes de ϕ_2^4 et ϕ_3^4 on ne peut bien sûr espérer qu'un programme différent, partant de l'étude de la série perturbative, aboutisse rapidement ou facilement. Cependant cette remarque même peut justifier de consacrer un certain temps à une approche différente, au cours de laquelle il est d'autre part très possible que des résultats intéréssants du point de vue de la simple théorie perturbative puissent être obtenus.

Nous conclurons par une remarque concernant le passage ultérieur des séries perturbatives en λ, m_0 aux séries fonctions des paramétres physiques. Si les fonctions de Schwinger et la densité d'action ont pu être définies comme fonctions analytiques de ν, λ, et m_0 , on peut passer aux variables ν, g , et $m_{phys.}$ Les résultats de Eckmann, Epstein, et de Eckmann, Epstein et Fröhlich [15] montrent que ces nouveaux paramétres sont fonctions C^∞ des premiers à ν = 2 et 3 ; et que la masse physique est sommable de Borel ce qui nous encourage à penser que ceci est vrai plus généralement et que le passage aux variables physiques n'altérera pas l'éventuelle sommabilité de Borel des séries interpolées que nous avons considérées.

Depuis la rédaction de cet article, la généralisation de ses résultats à tout le domaine Re $\nu < 4$ a été achevée par V. Rivasseau et E. Speer (à paraitre) .

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The Borel Transform in Euclidean φ_ν^4 Local Existence for $\mathrm{Re}\,\nu < 4$

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Abstract. We consider the φ^4 theory in Euclidean space of complex dimension ν and prove that, for $\mathrm{Re}\,\nu < 4$ the renormalized Feynman amplitudes grow at worst exponentially in the number of vertices in the graph. This implies that the Borel transform of any Schwinger function may be defined in a neighborhood of the origin in the Borel plane.

1. In this paper we prove the existence, in a neighborhood of the origin, of the Borel transform of the perturbative series for any Schwinger function of the Euclidean φ_ν^4 model, when the (possibly complex) dimension ν of space time satisfies $\mathrm{Re}\,\nu < 4$.

We do not discuss the more difficult problem of extending the domain of analyticity of the transform and proving the Borel summability of the theory (a problem which has been solved by constructive quantum field theory in the integer dimensions $\nu = 1, 2, 3$). A discussion of the background and motivation for this study is given in [1] (see also [2]), to which we refer the interested reader: here we summarize the problem briefly.

We consider then a truncated Schwinger function of $N \geqq 2$ fields. Such a function has a formal power series development in the coupling constant λ:

$$\delta\left(\sum_1^N p_i\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a_n(p, \nu),$$

where a_n is a sum of Feynman amplitudes which are associated with connected Feynman graphs and in which ν appears as a parameter. Since the number of such graphs is obviously smaller than the total number of graphs appearing in the development of the full Schwinger function, and since this number $K_{N,n} = (4n+N-1)!!$ is certainly bounded by $(n!)^2 \cdot (16)^n\ (4n+N-1)^{N/2}$, the con-

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vergence of the series for the Borel transform $B(t)$:

$$B(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} a_n(p, v)$$

in a neighborhood of the origin ($|t| < t_0$) will thus follow from our main result:

Theorem. *Let G_0 be a connected graph in the φ_v^4 theory with n internal and $N \geqq 2$ external vertices, and let $\delta(\sum p_i) I^R_{G_0}(p, v)$ be a renormalized Feynman amplitude for G_0. Then for $0 < \mathrm{Re}\, v < 4$, there is a constant A_v independent of n and of the graph G_0, such that:*

$$|I^R_{G_0}(v)| \leqq A_v^n. \tag{1}$$

Moreover, A_v may be chosen uniformly on compact subsets of $\{0 < \mathrm{Re}\, v < 4\}$.

We note although we do not deal here with this question, that a precise numerical calculation of A_v would provide a rigorous bound on the behavior of large orders of perturbation theory for φ_v^4, that could be compared with the more precise behavior obtained by Lipatov's method [3, 4].

We remark also (see [1]) that it suffices to establish the theorem for any one choice of renormalization method, since a finite renormalization will not affect the estimate (1). The generalization of the theorem to the region $\mathrm{Re}\, v < 4$ is trivial (see [1]).

The remainder of the paper is organized as follows. In Sect. 2 we introduce our notation, recall briefly the method of dimensional renormalization, and reduce the proof of the theorem above to an estimate on unrenormalized amplitudes. Section 3 is devoted to two preliminary estimates, on convergent graphs with generalized propagators, and on self-energy graphs. Section 4 gives an integral representation of an arbitrary amplitude in terms of the amplitudes considered in Sect. 3 and proves the final estimate.

2. We establish our notation for an arbitrary connected Feynman graph G in which line l has generalized propagator $(p^2 + \zeta_l)^{-\lambda_l}$, with $\lambda_l \geqq 1$, $\mathrm{Re}\, \zeta_l > 0$.

Let L_G, n_G, and h_G denote the number of lines, vertices, and loops, respectively, and write

$$\delta_G(v) = v/2 \cdot h_G - \sum_{l \in G} \lambda_l$$

for the superficial divergence in dimension v. Some vertices $\{v_a\}$ of G are external and formally associated to them are Euclidean momenta $\{p_a\}$ satisfying overall momentum conservation; the complex dimensional amplitude is a function of the invariants $\{s_{ab}\}$ formally given by $s_{ab} = p_a \cdot p_b$. The standard combinatoric functions are:

$$\begin{aligned} M_G(\alpha, \zeta) &= \sum \alpha_l \zeta_l \\ U_G(\alpha) &= \sum_T \prod_{l \notin T} \alpha_l \\ V_G(\alpha, s) &= U_G^{-1}(\alpha) \sum_{T_2} \Big(\prod_{l \notin T_2} \alpha_l \Big) \Big(\sum_{a \in E} p_a \Big)^2, \end{aligned} \tag{2}$$

where T is a (spanning) tree in G, T_2 a 2-tree which separates the external vertices into two non-empty sets, one of which is E. Finally the unrenormalized Feynman amplitude [5] of G is, omitting irrelevant factors:

$$I_G(\nu,s)=\int_0^\infty\dots\int_0^\infty\prod_{l\in G}[\Gamma(\lambda_l)^{-1}\alpha_l^{\lambda_l-1}d\alpha_l]\,U_G^{-\nu/2}\exp[-(V_G+M_G)]. \tag{3}$$

This integral converges absolutely for $\mathrm{Re}\,\nu<2$.

To study renormalization of the φ^4 theory, we fix $\nu_0<4$ and restrict ν to the domain $\Omega=\{\nu/0<\mathrm{Re}\,\nu<\nu_0\}$.

A graph G_0 in the φ^4 theory has $\zeta_l=m^2$, $\lambda_l=1$ for each line l; the amplitude I_{G_0} is meromorphic in ν and is analytic in the domain $\Omega'=\left\{\nu/0<\mathrm{Re}\,\nu<\nu_0,\ \nu\neq 4-\frac{2}{n}\right.$ for any $\left.n\geqq 1\right\}$ [5]. A one-particle irreducible (1PI) graph G_0 is superficially divergent for some $\nu\in\Omega$ [i.e. $\delta_G(\nu_0)>0$] if and only if G_0 has two external lines (i.e. is a self energy graph, obtained by amputating a graph in the Schwinger function of two fields) and satisfies $n_{G_0}<[2-(\nu_0/2)]^{-1}$.

Thus there is a finite collection $\mathscr{J}$ of divergent 1PI self energy graphs and all other graphs are superficially convergent.

Suppose then that G_0 is superficially convergent in $\Omega(\delta_{G_0}(\nu_0)\leqq 0)$. G_0 contains a (possibly empty) collection $\{H_1,\dots,H_I\}$ of divergent connected subgraphs, each isomorphic to an element of $\mathscr{J}$. Then the dimensionally renormalized amplitude $I^R_{G_0}$ has the form [5]:

$$I^R_{G_0}(\nu;s)=\sum_S\left[\prod_{i\in s}f_{H_i}(\nu)\right]I_{G_S}(\nu,s), \tag{4}$$

where the sum is over subsets $S\subset\{1,\dots,I\}$ such that $\{H_i,i\in S\}$ are disjoint, G_S is obtained from G by replacing each H_i, $i\in S$, by a single φ^2 vertex, and $f_{H_i}(\nu)$ depends only on the structure of H_i and is analytic in Ω'. Moreover, $I^R_{G_0}$ is analytic in Ω.

Now in Sect. 4 we will prove:

Lemma 1. *For any compact subset $K\subset\Omega'$ there is a constant $B_K\geqq 1$ such that, for $\nu\in K$ and for any Feynman graph G containing φ^4 and φ^2 vertices and at least two external lines:*

$$|I_G(\nu,s)|\leqq B_K^{L_G}. \tag{5}$$

Here we note that the main theorem of Sect. 1 follows immediately. For the right side of (4) contains fewer than 2^{L_G} terms; thus if $K\subset\Omega'$ is compact and $b_K=\max\limits_{H\in\mathscr{J}}\left\{\sup\limits_{\nu\in K}f_H(\nu)\right\}$ then for $\nu\in K$, $|I^R_G(\nu,s)|\leqq(2b_KB_K)^{L_G}$.

A similar bound follows on a compact subset $\bar{K}\subset\Omega$ by writing I^R_G in terms of Cauchy's integral formula on a Jordan curve in Ω' enclosing $\bar{K}$. Since $L_G<2n_G$, (1) follows.

3. We need two preliminary estimates for the proof of Lemma 1. The first controls the behavior of *convergent* graphs; the key idea is to use a convexity argument (as in [1, 2]) together with a standard result from linear programming to estimate $U_G^{-\nu/2}$ as a product.

Lemma 2. *For any compact $K \subset \Omega$ and any $a>0$, there is a constant $C_{K,a} \geqq 1$ such that, if $\bar{G}$ is a connected Feynman graph whose propagators $(p^2+\zeta_l)^{-\lambda_l}$ satisfy $\lambda_l \geqq 1$, $\mathrm{Re}\,\zeta_l \geqq a$, and for which $\delta_J(v_0) \leqq 0$ for every subgraph J, then for $v \in K$:*

$$|I_{\bar{G}}(v,s)| \leqq (C_{K,a})^{\sum_{l \in \bar{G}} \lambda_l}. \tag{6}$$

Proof. Let $\mu = \mathrm{Re}\, v$. We claim that there exist weights W_T for the (spanning) trees T of $\bar{G}$ such that:

$$\begin{aligned} &W_T \geqq 0 \\ &\sum W_T = 1 \\ &\sum_{T \ni l} W_T \leqq \frac{2\lambda_l}{\nu_0} \quad \forall l \in \bar{G}. \end{aligned} \tag{7}$$

Suppose this is true; then:

$$U_{\bar{G}}(\alpha) \geqq \sum_T W_T \prod_{l \notin T} \alpha_l \geqq \prod_T \Big(\prod_{l \notin T} \alpha_l \Big)^{W_T} = \prod_l \alpha_l^{\left(\sum_{T \ni l} W_T\right)} \tag{8}$$

by the standard inequality between geometric and arithmetic means. Hence from (3), since $V_{\bar{G}} \geqq 0$,

$$\begin{aligned} |I_{\bar{G}}| &\leqq \int_0^\infty \dots \int_0^\infty \prod_l [\Gamma(\lambda_l)^{-1} \alpha_l^{\lambda_l - 1} d\alpha_l] U_{\bar{G}}^{-\mu/2} \exp[-a \textstyle\sum \alpha_l] \\ &\leqq \prod_l \left[\int_0^\infty \Gamma(\lambda_l)^{-1} \alpha_l^{\varrho_l - 1} \exp(-a\alpha_l) d\alpha_l \right] \\ &= \prod_l \{ a^{\varrho_l} \Gamma(\varrho_l) / \Gamma(\lambda_l) \}, \end{aligned}$$

where $\varrho_l = \lambda_l - \frac{1}{2}\mu \sum_{T \ni l} W_T$ satisfies $\lambda_l \geqq \varrho_l \geqq 1 - \mu/\nu_0$.

Then (6) follows: for example, we may take $C_{K,a} = (\max\{1, a\}) \Gamma(r)$ where $r = \inf_{v \in K} \left(1 - \dfrac{\mathrm{Re}\, v}{\nu_0}\right)$.

It remains to prove the claim. Now (7) may be reformulated as follows: there exist weights $\{W_T\}$ and $\{u_l | l \in \bar{G}\}$ such that:

$$\begin{aligned} &W_T \geqq 0, \quad u_l \geqq 0 \\ &\sum W_T = 1 \\ &\sum_{T \ni l} W_T - u_l = 1 - 2\frac{\lambda_l}{\nu_0} \equiv \chi_l \quad \forall l \in \bar{G}. \end{aligned} \tag{9}$$

By a lemma of Farkas [6, 7], (9) has a solution if there is no solution (y, x_l) to the dual problem:

$$\begin{aligned} x_l &\leqq 0 \quad \forall l \in G \\ y+\sum_{l\in T} x_l &\geqq 0 \quad \forall T \\ y+\sum_{l\in \bar{G}} \chi_l x_l &< 0. \end{aligned} \tag{10}$$

Suppose then that a solution of (10) exists, we number the lines of $\bar{G}$ so that $x_{l_1} \leqq x_{l_2} \leqq \ldots \leqq x_{l_L} \leqq x_{l_{L+1}} \equiv 0$ and let J_k be the graph formed by $l_1, \ldots, l_k$.

The condition $\delta_{J_k}(v_0) \leqq 0$ is precisely:

$$\sum_{j=1}^{k} \chi_{l_j} \leqq L_{J_k} - h_{J_k} = n_{J_k} - c_{J_k}, \tag{11}$$

where c_{J_k} is the number of connected components of J_k.

Multiply (11) by $x_{l_{k+1}} - x_{l_k}$ and sum over $k=1, \ldots, L$ to obtain:

$$\sum_{\bar{G}} \chi_l x_l \geqq \sum_{k=1}^{L} x_{l_k}[(n_{J_k}-c_{J_k})-(n_{J_{k-1}}-c_{J_{k-1}})] = \sum_{l\in T} x_l, \tag{12}$$

where $T=\{l_k / n_{J_k}-c_{J_k} \neq n_{J_{k-1}}-c_{J_{k-1}}\}$ (set $n_{J_0}=0$, $c_{J_0}=0$, by convention).

But T is clearly a tree in $\bar{G}$, so that (10) implies:

$$\sum_{T} x_l \geqq -y > \sum_{\bar{G}} \chi_l x_l$$

contradicting (12). Thus (10) has no solution, and a solution exists for (9). □

The second estimate we need controls the high energy behavior of self energy graphs. Such results are known in greater generality [8] and we give a proof in our somewhat different case primarily for completeness.

For a self energy graph we write z for the external energy ($z=p^2$) and the notation $V(\alpha, z)=v(\alpha)z$.

Lemma 3. *Let H be a* 1-PI *self energy graph in the φ^4 theory. Then for* $\mathrm{Re}\, z > -\dfrac{m^2}{2}$, *the defining integral:*

$$I(v, z) = \int_0^\infty \ldots \int_0^\infty \prod d\alpha_l U^{-v/2} \exp[-(vz+(\textstyle\sum \alpha) m^2)] \tag{13}$$

converges absolutely if $\mathrm{Re}\, v<2$, *may be analytically continued to the domain Ω', and on compact subsets $K \subset \Omega'$ satisfies*

$$|I(v, z)| \leqq D_K |z+m^2|^{\bar{\delta}} \tag{14}$$

for some constant D_K, with $\bar{\delta}=\max\{\mathrm{Re}\,\delta_H(v), 0\}$.

Proof. We leave the case $v(\alpha)=0$ (in which the external vertices of H coincide) to the reader. In the general case we first note that, from (2), $v(\alpha) \leqq \sum_H \alpha_l$ and hence:

$$\mathrm{Re}\left[vz+\left(\sum_H \alpha_l\right) m^2\right] \geqq \tfrac{1}{2}\left(\sum_H \alpha_l\right) m^2. \tag{15}$$

Now following [5], we write $I=\sum I_\xi$ where the sum is over all s-families ξ for H, and in I_ξ introduce scaling variables $\{t_J | J\in\xi\}$ by $\alpha_l=\prod_{J\ni l} t_J$. Then:

$$I_\xi(v,z)=\int_0^\infty t_H^{-\delta_H(v)-1}dt_H\int_0^1\dots\int_0^1\prod_{\substack{J\in\xi\\J\neq H}} t_J^{-\delta_J(v)-1}dt_J e(t)^{-v/2}$$
$$\cdot\exp\left\{-t_H\left[\left(\prod_{J\in\xi'}t_J\right)f(t)z+g(t)m^2\right]\right\}, \tag{16}$$

where $c(t)$, $f(t)$, and $g(t)$ are independent of t_H, continuous and positive in the integration region, and $\xi'=\{J\in\xi/J\neq H$ and J connects the external vertices of $H\}$. The absolute convergence of (16) for $\mathrm{Re}\,v<2$ now follows from (15) and the inequality $\mathrm{Re}\,\delta_J(v)<0$, $J\in\xi$.

To continue I_ξ to all of Ω' we use the identity, valid for $\mathrm{Re}\,v<2$,

$$\int_0^1 t_J^{-\delta_J(v)-1}\varphi(t_J)dt_J=(-\delta_J(v))^{-1}\varphi(o)+\int_0^1\int_0^1 t_J^{-\delta_J(v)-1}dt_J\,d\tau_J\left[\frac{d}{d\tau_J}\varphi(t_J\tau_J)\right] \tag{17}$$

for each $J\in\mathfrak{J}=\{J\in\xi/J\neq H,\ \delta_J(v_0)>0\}$ carry out the τ_J derivatives, then do the t_H integral explicitly. The result is a finite sum of terms which we label by a partition $\mathfrak{J}=\mathfrak{J}'\cup\mathfrak{J}''$ describing which term of (17) occurs for each J, and non negative integers c,d, describing the number of times the z and m^2 terms respectively are pulled down from the exponential in (16) by τ derivatives. Thus (using $\mathfrak{J}\cap\xi'=\emptyset$):

$$I_\xi=\sum_{\mathfrak{J}',c,d}\Gamma(c+d-\delta_H(v))\prod_{\mathfrak{J}'}[-\delta_J(v)]^{-1}\int_0^1\dots\int_0^1\prod_{\mathfrak{J}''}t_J^{-\delta_J(v)}$$
$$\cdot dt_J d\tau_J\prod_{J\notin\mathfrak{J},J\neq H}t_J^{-\delta_J(v)-1}dt_J\left(z\prod_{\xi'}t_J\right)^c(m^2)^d R(t,\tau)S(t,\tau,z)^{\delta_H(v)-c-d},$$

where R is a continuous function and:

$$S(t,\tau,z)=\left[\prod_{\xi'}t_J f(t)z+g(t)m^2\right]\Big|_{\substack{t_J\to 0\,J\in\mathfrak{J}'\\ t_J\to\tau_J t_J,\,J\in\mathfrak{J}''}}$$
$$=\left[\prod_{\xi'}t_J\bar f(t,\tau)z+\bar g(t,\tau)m^2\right].$$

By (15), $\mathrm{Re}\,S\geqq\bar g(t,\tau)\dfrac{m^2}{2}$; hence the integral in (18) is convergent for $v\in\Omega$ and (18) provides an analytic continuation of I to Ω'.

To verify (14), note that for appropriate C_1, C_2:

$$C_1|z+m^2|\geqq|S|\geqq C_2\left[\prod_{\xi'}t_J|z|+m^2\right].$$

Hence since $|\arg S|\leqq\pi/2$,

$$|S^{\delta_H(v)-c-d}|=|S^{\delta_H(v)}||S^{-c}||S^{-d}|$$
$$\leqq e^{\pi/2|\mathrm{Im}\,\delta_H(v)|}\left[C_2\prod_{\xi'}t_J|z|\right]^{-c}[C_2m^2]^{-d}$$
$$\cdot\begin{cases}[C_1|z+m^2|]^{\mathrm{Re}\,\delta_H(v)} & \text{if}\quad \mathrm{Re}\,\delta_H(v)\geqq 0\\ [C_2m^2]^{\mathrm{Re}\,\delta_H(v)} & \text{if}\quad \mathrm{Re}\,\delta_H(v)\leqq 0.\end{cases}$$

Inserting these estimates in (18) yields (14).

4. Our goal in this section is to prove Lemma 1.

Thus, let G be a graph with φ^4 or φ^2 vertices and propagators $(p^2+m^2)^{-1}$; since Lemma 1 is an asymptotic result we may assume $\delta_G(v_0) \leqq 0$.

We identify in G a collection $G_1, \ldots, G_p$ of subgraphs as in Fig. 1; G_j is a maximal chain of $r(j)$ subgraphs $G_{jk} \in \mathscr{J}$, joined by lines $l_{j0}, l_{j1}, \ldots, l_{jr(j)}$.

Moreover, we suppose that G_{jk} is a maximal subgraph of G belonging to $\mathscr{F}$ and that every such maximal subgraph is a G_{jk} for some j, k. Feynman parameters in G_j are denoted α_{jk} (for l_{jk}) or α_{jki} (in G_{jk}); other Feynman parameters in G are denoted γ_i.

Fig. 1. The subgraphs G_j

We consider also the graph $\bar{G}$ obtained by replacing G_j in G with a single line l_j having Feynman parameters β_j and propagators $(p^2+\zeta_j)^{-\lambda_j}$ where

$$\lambda_j = -\delta_{G_j}(v_0) = 1 + \sum_{k=1}^{r(j)} (1-\delta_{G_{jk}}(v_0)). \tag{19}$$

Note that if J is a subgraph of $\bar{G}$ and J' the corresponding subgraph of G obtained by replacing l_j by G_j throughout, then $\delta_J(v_0) \leqq \delta_{J'}(v_0)$; thus Lemma 2 applies to $\bar{G}$. We will write $U, \bar{U}, U_j$ etc. ... for $U_G, U_{\bar{G}}, U_{G_j}$ etc. ... and from (2) note that:

$$\begin{aligned} U(\alpha,\gamma) &= \bar{U}(\beta,\gamma)/\beta_j = v_j \prod_1^p U_j \\ V(\alpha,\gamma,s) &= \bar{V}(\beta,\gamma,s)/\beta_j = v_j. \end{aligned} \tag{20}$$

Also:

$$\begin{aligned} U_j(\alpha) &= \prod_{k=1}^{r(j)} U_{jk}(\alpha) \\ v_j(\alpha) &= \sum_{k=1}^{r(j)} v_{jk}(\alpha) + \sum_{k=0}^{r(j)} \alpha_{jk} \end{aligned} \tag{21}$$

and hence:

$$I_{G_j}(v,z) = (z+m^2)^{-r(j)-1} \prod_{k=1}^{r(j)} I_{G_{jk}}(v,z). \tag{22}$$

We now give a purely formal derivation of a new representation of I_G; a similar representation is used in [9]. From (3) and (20),

$$\begin{aligned} I_G(v,s) = \int_0^\infty \ldots \int_0^\infty \prod d\alpha d\gamma \prod_{j=1}^p \{\delta(\beta_j - v_j(\alpha)) \\ \cdot [\beta_j/v_j(\alpha)]^{\lambda_j - 1} U_j^{-\nu/2} d\beta_j\} \bar{U}^{-\nu/2} \exp[-(\bar{V}+M)]. \end{aligned} \tag{23}$$

Now write

$$\delta(\beta_j - v_j) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\zeta_j = m^2/2} d\zeta_j \exp[\zeta_j(v_j - \beta_j)], \tag{24}$$

$$v_j^{-(\lambda_j - 1)} = \Gamma(\lambda_j - 1)^{-1} \int_0^\infty d\eta_j \eta_j^{\lambda_j - 2} \exp(-\eta_j v_j) \tag{25}$$

so that (23) becomes [using (3) for $\bar{G}$ and G_j, then (22)]:

$$I_G(v,s) = \int_{\mathrm{Re}\,z_j > -m^2/2} \prod_{j=1}^{P} d\mu_j(z_j) I_{\bar{G}}(v,s) \prod_{j=1}^{P} \left\{ (z_j + m^2)^{-r(j)-1} \prod_k I_{G_{jk}}(v, z_j) \right\}, \tag{26}$$

where $z_j = \eta_j - \zeta_j$ and $d\mu_j(z_j) = \frac{1}{2\pi i}(\lambda_j - 1)\eta_j^{\lambda_j - 2} d\eta_j d\zeta_j$ [recall that $I_{\bar{G}}$ contains a factor $\prod_j \Gamma(\lambda_j)^{-1}$].

Lemma 4. *The formula* (26) *is valid for all* $v \in \Omega'$.

Proof. We first establish (26) for $\mathrm{Re}\,v$ sufficiently small. Let φ be a smooth, even, non-negative function of compact support on $\mathbb{R}$ for which $\int_{\mathbb{R}} \varphi(x)dx = 1$, and let $\tilde{\varphi}(t) = \int_{\mathbb{R}} \varphi(x) e^{ixt} dx$. For $\varepsilon > 0$, we may regularize the δ-function above by inserting a factor $\tilde{\varphi}(\varepsilon\,\mathrm{Im}\,\zeta_j)$ in the integral (24); substitution of (25) and the modified (24) into (23) leads us to define

$$I^\varepsilon(v,s) = \int_{\mathrm{Re}\,z_j > -m^2/2} \prod_{j=1}^{P} \tilde{\varphi}(\varepsilon\,\mathrm{Im}\,z_j) \Gamma(\lambda_j)^{-1} d\mu_j(z_j) \int_0^\infty \prod d\alpha\, d\gamma$$
$$\cdot \prod_{j=1}^{P} \beta_j^{\lambda_j - 1} d\beta_j [\bar{U}_{(\beta,\gamma)}]^{-\nu/2} \exp[-(\bar{V} + \bar{M})] \prod_{j=1}^{P} U_j(\alpha)^{-1/2} \exp[-(v_j z_j + M_j)]. \tag{27}$$

This integral is absolutely convergent, since taking the absolute value of the integrand replaces v by $\mathrm{Re}\,v$ and ζ_j by $m^2/2$, thus

$$\int |\ldots| d\eta\, d\zeta\, d\alpha\, d\gamma\, d\beta = \int \prod_j |\tilde{\varphi}(\varepsilon\,\mathrm{Im}\,z_j)|\, d\mu_j(z_j) I_{\bar{G}}(\mathrm{Re}\,v, s)|_{\zeta_j = m^2/2}$$
$$\cdot \prod_j I_{G_j}(\mathrm{Re}\,v, \mathrm{Re}\,z_j)$$

which converges by (22) and Lemmas 2 and 3.

Thus we may evaluate (27) by doing the α, β, and γ integrals to obtain

$$I^\varepsilon(v,s) = \int \prod_j \tilde{\varphi}(\varepsilon\,\mathrm{Im}\,z_j) d\mu_j(z_j) I_{\bar{G}}(v,s) \prod_j [(z_j + m^2)^{-r(j)-1} \prod_k I_{G_{jk}}(v, z_j)]. \tag{28}$$

Since $\tilde{\varphi}(o) = 1$ the Lebesgue dominated convergence theorem and Lemmas 2, 3 yield:

$$\lim_{\varepsilon \to 0^+} I^\varepsilon(v,s) = [\text{right hand side of (26)}].$$

[The use of Lemmas 2 and 3 to estimate integrals such as (28) is given in detail in the proof of Lemma 1, below.]

It remains to show that

$$\lim_{\varepsilon\to 0^+} I^\varepsilon(v,s)=I_G(v,s). \tag{29}$$

For this, we do the z integrals in (27) to obtain

$$I^\varepsilon(v,s)=\int\prod d\alpha\prod d\gamma\prod_{j=1}^{P}[v_j^{-(\lambda_j-1)}U_j^{-\nu/2}e^{v_jm^2/2}]\exp(-M)G^\varepsilon(\alpha,\gamma). \tag{30}$$

where $G^\varepsilon(\alpha,\gamma)=\int_0^\infty\prod_j\{\beta_j^{\lambda_j-1}\varepsilon^{-1}\varphi[(\beta_j-v_j)/\varepsilon]\exp(-\beta_jm^2/2)d\beta_j\}\bar U^{-\nu/2}\exp(-\bar V)$.

Clearly

$$\lim_{\varepsilon\to 0^+} G^\varepsilon(\alpha,\gamma)=(v_j)^{\lambda_j-1}\exp(-v_jm^2/2)[\bar U^{-\nu/2}\exp(-\bar V)]|_{\beta_j=v_j} \tag{31}$$

and if we can take this limit outside the integral in (30), (29) is established. Now choosing any term in $\bar U$ we may estimate

$$\bar U\geqq\prod\beta_j^{b_j}\prod\gamma_i^{c_i},$$

where b_j, c_i are zero or one; using $\bar V\geqq 0$ we have

$$|G^\varepsilon|\leqq\prod_i\gamma_i^{-c_i\mathrm{Re}\nu/2}\prod_j\left\{\sup_{\beta_j\geqq 0}\beta_j^{\lambda_j-1-b_j\mathrm{Re}\nu/2}\exp\left(-\beta_j\frac{m^2}{2}\right)\right\}. \tag{32}$$

Since $\lambda_j>1$ the supremum is finite for Reν sufficiently small. From (21),

$$v_j^{-\lambda_j-1}\leqq\left(\sum_{k=1}^{r(j)}\alpha_{jk}\right)^{\sum_k(\delta_{G_{jk}}(v_0)-1)}\leqq\prod_{k=1}^{r(j)}\alpha_{jk}^{(\delta_{G_{jk}}(v_0)-1)}. \tag{33}$$

Using (32), (33), and (15), we see that the integrand in (30) is in absolute value less than

$$(\mathrm{const})\cdot\prod_i[\gamma_i^{-c_i\mathrm{Re}\nu/2}e^{-\gamma_im^2}]\prod_j\left\{e^{-\alpha_{j0}m^2}\prod_{k=1}^{r(j)}[\alpha_{jk}^{(\delta_{G_{jk}}(v_0)-1)}e^{-\alpha_{jk}m^2}]\right.$$

$$\left.\cdot\prod U_{jk}^{-\nu/2}e^{-1/2M_{jk}}\right\}$$

and for Reν small the dominated convergence theorem yields (29).

To complete the proof we must verify (26) for all $\nu\in\Omega'$. But by Lemmas 2 and 3 the right hand side is absolutely convergent for all $\nu\in\Omega'$, (again see the proof of Lemma 1, below) and hence analytic in Ω'. The result follows by the principle of analytic continuation.

Proof of Lemma 1. We apply Lemmas 2 and 3 to estimate (26) for $\nu\in K\subset\Omega'$, K compact. Let $L'=L_G-\sum_j L_{G_j}=L_{\bar G}-P$ and set $t_j=\mathrm{Im}\,z_j$.

Then

$$|I_G(v,s)| \leqq C_{K,m^2/2}^{(L'+\sum\lambda_j)} \prod_{j=1}^{P} \left\{ \frac{D_K(\lambda_j-1)}{2\pi} \int_0^\infty \eta_j^{\lambda_j-2}\, d\eta_j \int_{-\infty}^{+\infty} dt_j |z_j+m^2|^{\lambda_j-\mu_j} \right\} \tag{34}$$

with $\mu_j = r(j)+1-\sum_k \max[0, \mathrm{Re}\,\delta_{G_{jk}}(v)]$.

We introduce polar coordinates $\eta_j = r_j \sin\theta_j$, $t_j = r_j \cos\theta_j$, and note

$$|z_j+m^2| = |\eta_j+it_j+m^2/2| \geqq A(r_j+1)$$

for an appropriate constant A.

Thus we estimate the bracketed term in (34) by:

$$\{\ \} \leqq D_K \frac{(\lambda_j-1)}{2\pi} 2 \int_0^{\pi/2} (\sin\theta_j)^{\lambda_j-2}\, d\theta_j \int_0^\infty r_j^{\lambda_j-1} [A(r_j+1)]^{-\mu_j}\, dr_j$$

$$\leqq \frac{D_K}{\pi} \left(\frac{\pi}{2}\right)^{\lambda_j-1} A^{-\mu_j} B(\lambda_j, \mu_j-\lambda_j),$$

where B is the beta function. (We have used $\sin\theta_j \leqq \theta_j$.)

Since $\lambda_j > 1$ and $\mu_j - \lambda_j$ is uniformly bounded away from zero on K [see (19)], the beta function is uniformly bounded by some K-dependent constant.

Thus (34) becomes:

$$|I_G(v,s)| \leqq (C_{K,m^2/2})^{(L'+\sum\lambda_j)} \left(\frac{\pi}{2}\right)^{\sum(\lambda_j-1)} A^{-\sum\mu_j} E_K^P\ ;$$

since $L' \leqq L_G$, $\sum\lambda_j \leqq \sum[r(j)+1] \leqq L_G$ by (19), $P < L_G$ and $\sum\mu_j \leqq \sum[r(j)+1] \leqq L_G$, this yields a bound of the form (5).

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Local Existence of the Borel Transform in Euclidean Φ_4^4

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Abstract. We bound rigorously the large order behaviour of Φ_4^4 euclidean perturbative quantum field theory, as the simplest example of renormalizable, but non-super-renormalizable theory. The needed methods are developed to take into account the structure of renormalization, which plays a crucial role in the estimates. As a main theorem, it is shown that the Schwinger functions at order n are bounded by $K^n n!$, which implies a finite radius of convergence for the Borel transform of the perturbation series.

I. Introduction

In this work we give rigorous bounds on the Feynman amplitudes at large order for renormalized Φ_4^4 euclidean quantum field theory. We prove the "local existence" of the Borel transform of its perturbative series for the connected Green's functions (or Schwinger functions): the Borel transformed perturbative series is shown to have a finite radius of convergence. Such a theorem is reached by finding direct estimates of the renormalized amplitudes related to Feynman graphs. We combine the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization scheme with the use of the so-called Hepp sectors to give bounds which depend on the renormalization structure of the graphs. Then we count the number of graphs with a given renormalization structure, and we can bound the term of order n of any Schwinger function by $K^n n!$, which proves our main result.

The same property was already known for Φ_ν^4 in the super-renormalizable domain $\mathrm{Re}\,\nu < 4$ [1]. Through Borel summability for integer dimensions $\nu = 2, 3$ [2], perturbative and nonperturbative methods of constructive quantum field theory are connected [3]. In this domain, each Feynman amplitude is uniformly bounded by K^n. Conversely for $\nu = 4$, such a bound cannot exist and the situation remains unresolved. Despite all efforts there has been almost no rigorous result. It has been remarked that some graphs of order n grow like $n!$ [4]. (Of course we recover this particular result and we show how few graphs are dangerous.) Heuristic arguments have been given, to make plausible the appearance of singularities, called renormalons, on the real positive axis of the Borel plane [5].

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That would destroy the usual Borel summability. However it was believed that the Borel transformed series has a finite radius of convergence; this has been evaluated by semi-classical approximations [6].

Since we prove rigorously this last fact, we hope to provide a new incitement to the study of renormalizable (but non-super-renormalizable) field theories. Starting from this result, one could try to perform an analytic continuation of the function we define in the Borel plane, or to use our methods for discovering a new constructive approach.

This paper is organized as follows: Sect. II is devoted to the definitions and main results. In Sect. III we define one of our main tools, the classification of forests, and we give the proof of the bound on individual amplitudes. Sect. IV gives a brief discussion on our results and their possible generalizations. Appendices A–C solve the graphical and combinatoric problems, very different from those encountered in the super-renormalizable case.

II. Definitions and Results

II.1. The Φ_4^4 Model

We consider an euclidean field theory of massive scalar self-interacting bosons in 4-dimensional space-time. The interaction is defined by

$$\mathscr{L}_I(\Phi) = -\lambda\Phi^4 + \text{counter-terms}.$$

Up to now this is a field theory only in the perturbative sense. We define the truncated Schwinger function of N fields, $N \geqq 2$, via a formal expansion in the coupling constant:

$$\mathbf{S}_N^T(\not{p}, \lambda) = \delta\left(\sum_{a=1}^{N} \not{p}_a\right) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \mathbf{a}_n^N(\not{p}), \tag{II.1}$$

where $\not{p} = (\not{p}_1, \ldots, \not{p}_N)$ is a set of external euclidean momenta, and $\mathbf{a}_n^N$ is a sum of renormalized Feynman amplitudes, associated to Feynman graphs with n internal vertices and N external legs. These graphs have exactly 4 lines attached at each vertex.

The unrenormalized Feynman amplitude I_G associated to the graph G is given in the α-parametric representation by the following formal integral:

$$I_G(\not{p}) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{\ell} d\alpha_i \left[\exp\left(-\mu^2 \sum_{i=1}^{\ell} \alpha_i\right)\right] Z_G(\not{p}, \alpha), \tag{II.2}$$

where a parameter α_i has been attached to each internal line $i = 1, \ldots, \ell$ of the graph G, and μ is the mass of the bosons, which we fix at the value 1 in the rest of the paper.

$$Z_G(\not{p}, \alpha) = 1/[U_G(\alpha)]^2 \exp[-V_G(\not{p}, \alpha)/U_G(\alpha)], \tag{II.3}$$

$$U_G(\alpha) = \sum_S \prod_{i \notin S} \alpha_i, \tag{II.4}$$

$$V_G(\not{p}, \alpha) = \sum_T \left(\prod_{i \notin T} \alpha_i\right)\left(\sum_{a \in T_1} \not{p}_a\right)^2. \tag{II.5}$$

U_G and V_G are the standard Symanzik polynomials: S runs over the spanning trees (or "one-trees") of G, T over the 2-trees of G which separate the external lines of G into two non-empty sets, one of which is T_1. It is also convenient to define:

$$|\not{k}| = 1 + N \sum_{a=1}^{N} \not{k}_a^2 . \tag{II.6}$$

This is not a norm in the usual sense, but we have:

$$V_G(\not{k}, \alpha) \leqq |\not{k}| \sum_T \prod_{i \notin T} \alpha_i . \tag{II.7}$$

Before renormalizing, hence defining a finite part of the integral (II.2), we introduce our definitions and notations for the graphs, used everywhere in the rest of the paper.

II.2. Graph Theory

Throughout this paper, the word graph means a "labeled" graph, hence a set of distinguished vertices with internal and external lines. For any graph G we define:

$n(G)$ as the number of vertices of G,
$\ell(G)$ as the number of internal lines of G,
$N(G)$ as the number of external lines of G,
$L(G)$ as the number of independent loops of G,
$c(G)$ as the number of connected components of G.

We have the topological relation

$$L(G) = \ell(G) - n(G) + c(G) . \tag{II.8}$$

Moreover if G appears in the expansion (II.1) it has exactly 4 lines attached at each vertex. Hence $N(G)$ is even, and

$$\ell(G) = 2n(G) - N(G)/2 . \tag{II.9}$$

The superficial degree of convergence of G is defined by:

$$\omega(G) = \ell(G) - 2L(G) . \tag{II.10}$$

From (II.8) and (II.9) we have also, if G is a connected graph of Φ_4^4

$$\omega(G) = N(G)/2 - 2 . \tag{II.11}$$

Definition II.1. A subgraph F of G is a set of internal lines of G, together with the corresponding attached vertices. An external vertex of F is a vertex attached to at least one external line of F.

We extend also the definitions of ℓ, n, L, N, c, ω to subgraphs in the natural way. $F \subset F'$ means that the set of internal lines defining F is strictly included in the set of internal lines defining F'. For non-strict inclusion we always use the symbol $\subseteq$.

Since we look at the expansion of truncated Schwinger functions with $N \geqq 2$, there will be no vacuum ($N = 0$) graphs or subgraphs in our problem. Moreover the graphs containing tadpoles (subgraphs with only one external vertex) will vanish in expansion (II.1) after renormalization (see Subsect. II.3).

Definition II.2. A subgraph F of G is said to be divergent if $\omega(F) \leqq 0$. Using (II.11) we remark that the connected divergent subgraphs F have $N(F)=4$ or $N(F)=2$. They will be called quadrupeds or bipeds, respectively.

Definition II.3. Two subgraphs F, F' are disjoint if they have no line and no vertex in common. They overlap if they are not disjoint and do not satisfy an inclusion relation ($F \subsetneqq F'$ or $F' \subsetneqq F$). A forest $\mathcal{F}$ is a set of non-overlapping connected subgraphs. F is said to be compatible with $\mathcal{F}$ if $\mathcal{F} \cup \{F\}$ is a forest.

For any $\mathcal{F}$ and F compatible with $\mathcal{F}$, we define

$$\mathcal{A}_{\mathcal{F}}(F) = \{F' \mid F' \in \mathcal{F} ; F' \subset F : \nexists F'' \in \mathcal{F}, F' \subset F'' \subset F\},$$

$$A_{\mathcal{F}}(F) = \bigcup_{F' \in \mathcal{A}_{\mathcal{F}}(F)} F' = \bigcup_{\substack{F' \in \mathcal{F} \\ F' \subset F}} F'.$$

$A_{\mathcal{F}}(F)$ may be empty and generally does not belong to $\mathcal{F}$, except if $\mathcal{A}_{\mathcal{F}}(F)$ has exactly one element. Conversely we denote $B_{\mathcal{F}}(F)$ the smallest subgraph in $\mathcal{F}$ which strictly includes F

$$B_{\mathcal{F}}(F) \in \mathcal{F} ; \quad B_{\mathcal{F}}(F) \supset F ; \quad \nexists F'' \in \mathcal{F}, \quad F \subset F'' \subset B_{\mathcal{F}}(F).$$

If there exists no such subgraph, we define $B_{\mathcal{F}}(F) = G$. If $F \subset F'$ we have the usual notion of reduced graph: F'/F is the graph obtained from F' when each connected component of F has been reduced to a single vertex. If F is compatible with $\mathcal{F}$ we define: $F/\mathcal{F} = F/A_{\mathcal{F}}(F)$. Finally for $F \in \mathcal{F}$ and any subgraph X, denote by $X \cap F/\mathcal{F}$ the subgraph $[(X \cup A_{\mathcal{F}}(F)) \cap F]/\mathcal{F}$.

Definition II.4. A subgraph F is said to be one line reducible (OLR) if there exists a line of F such that by its removal the number of connected components of F increases. A single line is OLR. A connected subgraph which is not OLR is called proper. A proper component of a subgraph F is a maximal proper subgraph $F' \subsetneqq F$.

Definition II.5. A quadruped Q is said to be open if it is proper, and if there exists a proper biped B such that $Q \subset B$, and that both external vertices of B are external vertices of Q. The biped B is then unique and is called the closure Q^* of Q. In other words an open quadruped Q is obtained from its closure $B = Q^*$ by removing a single line, or a chain of proper bipeds, as pictured in Fig. 1.

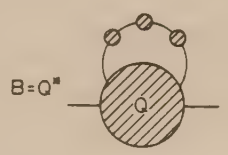

Fig. 1

A proper subgraph F is said to be closed if

$$\forall Q \text{ open quadruped}, \quad Q \subsetneqq F \Rightarrow Q^* \subsetneqq F. \tag{II.12}$$

Finally we define the closure F^* of any proper subgraph F as the smallest closed subgraph containing F. It is easy to verify that this extended definition coincides with the former one for open quadrupeds, that a proper biped is always closed and

that the closure preserves the inclusion relations:

$$\forall F_1, F_2 \text{ proper}, \quad F_1 \subseteq F_2 \Rightarrow F_1^* \subseteq F_2^*. \tag{II.13}$$

Moreover if F_1^* and F_2^* overlap, then F_1 and F_2 also overlap. A connected divergent subgraph which is not closed is called unessential. Then either F is proper and is an open quadruped Q (Fig. 1), or it is not proper (Fig. 2 for bipeds, or Fig. 3 for quadrupeds).

Fig. 2

Fig. 3

A forest of proper (respectively closed, divergent closed, etc. ...) subgraphs is called in brief a proper (respectively closed, divergent closed, etc. ...) forest. We remark that in the reduction of a graph by a closed divergent forest no tadpole may appear.

Definition II.6. For any divergent forest $\mathcal{F}$ we define:

$$f(\mathcal{F}) = q(\mathcal{F}) + 2b(\mathcal{F}), \tag{II.14}$$

where $q(\mathcal{F})$ and $b(\mathcal{F})$ are respectively the numbers of quadrupeds and of bipeds in $\mathcal{F}$. Then we define

$$f_1(G) = \sup_{\mathcal{F} \text{ closed divergent forests of } G} [f(\mathcal{F})], \tag{II.15}$$

$$f_2(G) = \begin{cases} \sup_{\mathcal{F} \text{ forests of quadrupeds of } G} [f(\mathcal{F})], & \text{if} \quad N(G) > 2, \\ 1 + \sup_{\mathcal{F} \text{ forests of quadrupeds of } G} [f(\mathcal{F})], & \text{if} \quad N(G) = 2. \end{cases} \tag{II.16}$$

Lemma II.1.

$$f_1(G) \leqq f_2(G). \tag{II.17}$$

Proof. For any forest $\mathcal{F}$ we call $\mathcal{B}(\mathcal{F})$ the subforest of its bipeds. Let $\mathcal{F}$ be a closed divergent forest, with $\mathcal{B}(\mathcal{F}) = \{B_1, \ldots, B_j, \ldots, B_{b(\mathcal{F})}\}$. We choose a line i_1 in $B_1 - A_{\mathcal{F}}(B_1)$ (which is not empty) and an external line i'_1 of B_1, internal in G (if $B_1 \neq G$). Then we define

$$Q(B_1) = B_1 - \{i_1\}; \qquad R(B_1) = B_1 \cup \{i'_1\}.$$

Any $F \in \mathcal{F}$ which overlaps with $R(B_1)$ is replaced by $F \cup R(B_1) = F^1$, and we define a new forest $\mathcal{F}_1$ by these replacements, by removing B_1 and by adding $Q(B_1)$ and $R(B_1)$ At this first step, since each F is proper, $F \cap R(B_1)$ is empty or is the vertex at the end of i'_1 which does not belong to B_1. In the second case $N(F \cup R(B_1)) = N(F)$ (see Figs. 2 or 3) and $\mathcal{F}_1$ satisfies the following conditions (H), which we take as

induction hypotheses at the step j; let us assume we have built a divergent forest $\mathscr{F}_j$ such that

$$\text{(H)}\begin{cases} f(\mathscr{F}_j)=f(\mathscr{F}); \quad b(\mathscr{F}_j)=b(\mathscr{F})-j; & \text{(H1)}\\ \forall F\in\mathscr{F} \text{ and } \forall Q \text{ non-closed proper component of } F, & \\ Q^*=B_k \text{ with } k\leqq j; & \text{(H2)}\\ \mathscr{B}(\mathscr{F}_j)=\{B^j_{j+1},\ldots,B^j_{j'},\ldots,B^j_{b(\mathscr{F})}\}\text{; and } F=B^j_{j'}\Leftrightarrow B_{j'} \text{ proper component of } F. & \text{(H3)}\end{cases}$$

Then defining i_{j+1}, i'_{j+1}, $Q(B^j_{j+1})$, $R(B^j_{j+1})$ in the same way as before, with $\mathscr{F}$ replaced by $\mathscr{F}_j$, it is easy to verify that

a) $i_{j+1}\in B_{j+1}$,

b) $Q(B^j_{j+1})\notin\mathscr{F}_j$ since it has a non-closed proper component whose closure is B_{j+1} [see (H2)],

c) $R(B^j_{j+1})\notin\mathscr{F}_j$ since it has B_{j+1} as a proper component [see (H3)],

d) If $F\in\mathscr{F}_j$ overlaps with $R(B^j_{j+1})$, $F\cap R(B^j_{j+1})$ is the vertex at the end of i'_{j+1} which does not belong to B^j_{j+1} [see again (H3)].

This allows us to perform the step $j+1$, and to build $\mathscr{F}_{j+1}$ still satisfying the conditions (H). Finally $\mathscr{F}_{b(\mathscr{F})}$ is a forest of quadrupeds with $f(\mathscr{F})$ elements, or possibly $f(\mathscr{F})-1$ if G is a biped (in which case $R(G)$ cannot be defined). In any case, with the definintion (II.16), (II.17) is proved.

Remark. In fact, given a forest $\mathscr{F}$ of quadrupeds, by considering the forest $\mathscr{F}^*$ of the closures of the proper components of the subgraphs in $\mathscr{F}$, one could show $f_1(G)=f_2(G)$. Since we do not use this equality, but only (II.17), we leave its verification to the reader.

The numbers $f_1(G)$ and $f_2(G)$ will be useful to state precisely the results of the next sections. It is proved in Appendix C that $f_2(G)\leqq n(G)$. Hence by (II.17) we have also $f_1(G)\leqq n(G)$.

II.3. Renormalization

Following [7] we define the usual BPHZ renormalized Feynman amplitude with subtractions at zero external momenta by an operator acting directly on the integrand Z_G in the α-parameters representation (II.2). By this subtraction procedure, the graphs with tadpoles subgraphs vanish.

For every connected subgraph F of G we define a Taylor operator $\mathscr{T}_F$ acting on Z_G in the following way: let T^r be the usual Taylor operator of order r; if $f(x)$ is $\mathscr{C}^\infty$ at $x=0$ we have

$$T^r f=0, \quad \text{if} \quad r<0,$$

$$T^r f=\sum_{s=0}^{r}\frac{x^s}{s!}f^{(s)}(0), \quad \text{if} \quad r\geqq 0.$$

Now if f is such that $x^m f$ is $\mathscr{C}^\infty$ at $x=0$ for some integer m, we define

$$\mathscr{T}^r_x f=x^{-m}T^{m+r}(x^m f).$$

This definition is independent of m. Let $Z_G(\varrho_F, \alpha)$ be obtained from $Z_G(\alpha)$ by scaling α_i into $\varrho_F \alpha_i$ for $i \in F$; then $\mathscr{T}_F$ is finally defined by

$$\mathscr{T}_F Z_G(\alpha) = [\mathscr{T}_{\varrho_F}^{-\ell(F)} Z_G(\varrho_F, \alpha)]_{\varrho_F = 1}. \tag{II.18}$$

The renormalized amplitude associated to G is then expressed by the following absolutely convergent integral representation

$$I_G^R(\not{p}) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{\ell} d\alpha_i \left[\exp\left(-\sum_{i=1}^{\ell} \alpha_i\right)\right] Z_G^R(\not{p}, \alpha). \tag{II.19}$$

$$Z_G^R(\not{p}, \alpha) = \mathscr{R}\{1/[U_G(\alpha)]^2 \exp[-V_G(\not{p}, \alpha)/U_G(\alpha)]\}. \tag{II.20}$$

The operator $\mathscr{R}$ may be defined in many equivalent ways, among which we select the following one [7]

$$\mathscr{R} = \sum_{\bar{\mathscr{F}}} \prod_{F \in \bar{\mathscr{F}}} (-\mathscr{T}_F), \tag{II.21}$$

where the sum is performed over all proper divergent forests of G, including the empty one. We note that Taylor operators of non-overlapping subgraphs commute, hence the products in (II.21) are taken in an arbitrary order.

It is however possible to simplify formulae (II.18) and (II.21) to obtain a more practical form of the renormalization, adapted to our specific Φ_4^4 model. We introduce the following notations, for F a subgraph of G: S_F (respectively T_F) is the set of spanning trees (respectively two-trees) whose restriction to F is a spanning tree of F; $S_{\bar{F}}$ (respectively $T_{\bar{F}}$) is the set of spanning trees (respectively two-trees) whose restriction to F is not a spanning tree of F.

U_F, $U_{\bar{F}}$, V_F, $V_{\bar{F}}$ are defined by formulae (II.4) and (II.5) with the sums respectively restricted to S_F, $S_{\bar{F}}$, T_F, $T_{\bar{F}}$. These notations help us to give explicitly the action of the Taylor operators:

i) if F is convergent,

$$\mathscr{T}_F Z_G = 0, \tag{II.22}$$

ii) if F is a quadruped, we have

$$U_G(\varrho_F, \alpha) = \varrho_F^{L(F)}[U_F(\alpha) + \varrho_F \tilde{U}_{\bar{F}}(\varrho_F, \alpha)],$$
$$V_G(\varrho_F, \not{p}, \alpha) = \varrho_F^{L(F)}[V_F(\not{p}, \alpha) + \varrho_F \tilde{V}_{\bar{F}}(\varrho_F, \not{p}, \alpha)],$$

where $\tilde{U}_{\bar{F}} = U_{\bar{F}}/\varrho_F$ and $\tilde{V}_{\bar{F}} = V_{\bar{F}}/\varrho_F$ are regular at $\varrho_F = 0$. Hence

$$\mathscr{T}_F Z_G = 1/[U_F(\alpha)]^2 \exp(-V_F(\not{p}, \alpha)/U_F(\alpha)) = \mathscr{T}_{\chi_F}^0 Z_G(\chi_F, \not{p}, \alpha), \tag{II.23}$$

with

$$Z_G(\chi_F, \not{p}, \alpha) = \frac{1}{[U_F + \chi_F U_{\bar{F}}]^2} \exp\left(-\frac{V_F + \chi_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right). \tag{II.24}$$

By Taylor's integral formula

$$(1 - \mathscr{T}_F) Z_G = \int_0^1 d\chi_F \frac{d}{d\chi_F} Z_G(\chi_F, \not{p}, \alpha)$$
$$= \int_0^1 d\chi_F [U_F + \chi_F U_{\bar{F}}]^{-3} \left[-2U_{\bar{F}} + \frac{U_{\bar{F}} V_F - U_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right] \exp\left(-\frac{V_F + \chi_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right). \tag{II.25}$$

iii) if F is a biped, we remark that a spanning tree or a two-tree which are not spanning trees in F have at most two connected components in F, each one containing one of the external vertices of F.

Hence

$$U_G(\varrho_F,\alpha)=\varrho_F^{L(F)}[U_F(\alpha)+\varrho_F U_{\bar F}(\alpha)],$$
$$V_G(\varrho_F,\not p,\alpha)=\varrho_F^{L(F)}[V_F(\not p,\alpha)+\varrho_F V_{\bar F}(\not p,\alpha)].$$

It follows that

$$\mathscr{T}_F Z_G=\mathscr{T}^1_{\chi_F} Z_G(\chi_F,\not p,\alpha)=\frac{1}{U_F^2}\left[1-2\frac{U_{\bar F}}{U_F}+\frac{U_{\bar F}V_F-U_F V_{\bar F}}{U_F^2}\right]\exp\left(-\frac{V_F}{U_F}\right), \quad \text{(II.26)}$$

and finally

$$\begin{aligned}(1-\mathscr{T}_F)Z_G&=\int_0^1 d\chi_F(1-\chi_F)\frac{d^2}{d\chi_F^2}Z_G(\chi_F,\not p,\alpha)\\&=\int_0^1 d\chi_F(1-\chi_F)[U_F+\chi_F U_{\bar F}]^{-4}\exp\left(-\frac{V_F+\chi_F V_{\bar F}}{U_F+\chi_F U_{\bar F}}\right)\\&\quad\cdot\left[6U_{\bar F}^2-6U_{\bar F}\frac{U_{\bar F}V_F-U_F V_{\bar F}}{U_F+\chi_F U_{\bar F}}+\left(\frac{U_{\bar F}V_F-U_F V_{\bar F}}{U_F+\chi_F U_{\bar F}}\right)^2\right]. \quad \text{(II.27)}\end{aligned}$$

In the whole $\mathscr{R}$ operation, further simplifications appear since in fact Taylor operators for unessential subgraphs are unessential: the connected divergent subgraphs which are not proper already do not appear in (II.21). But we will show also that open quadrupeds do not contribute to the $\mathscr{R}$ operation, which is not a result quoted in [7]. Before proving it, we state for non-closed subgraphs a more general lemma which is useful in Sect. III:

Lemma II.2. *Let F be a proper subgraph and F^* its closure. Then*

$$S_{\bar F^*}\subseteq S_{\bar F};\qquad T_{\bar F^*}\subseteq T_{\bar F}. \quad \text{(II.28)}$$

Proof. If F is an open quadruped, by definition (II.5) every external vertex of F^* is an external vertex of F. By induction, this remains true for any proper non-closed subgraph F, since F^* is obtained from F by adding some lines or chains of bipeds which have both ends in F.

Now let S be a one-tree (respectively T a two-tree) which is not a one-tree in F^*. Then $S\cap F^*$ (respectively $T\cap F^*$) has at least two connected components γ_1, γ_2, some of them possibly reduced to a single vertex, and each of them must contain at least one external vertex of F^*, say v_1, v_2. There is no path in $S\cap F^*$ (respectively in $T\cap F^*$) from v_1 to v_2, therefore no path in $S\cap F$ (respectively in $T\cap F$) from v_1 to v_2, which proves that S (respectively T) is not a one-tree in F.

Now from Lemma II.2 and formulae (II.23) and (II.27) it is easy to conclude that if F is an open quadruped,

$$\mathscr{T}_F(1-\mathscr{T}_{F^*})Z_G=0. \quad \text{(II.29)}$$

A definition of the $\mathscr{R}$ operator follows which is more convenient for the next sections:

Lemma II.3.

$$\mathscr{R} = \sum_{\mathscr{F}} \prod_{F \in \mathscr{F}} (-\mathscr{T}_F). \tag{II.30}$$

where the sum is performed over all (possibly empty) closed divergent forests of G.

Proof. Let us consider all forests $\mathscr{F}$ in which a given open quadruped F is maximal among the open quadrupeds of $\mathscr{F}$. Then F^* is a proper biped, which cannot overlap with any closed biped or quadruped (see Appendix B). Hence F^* cannot overlap with any subgraph of any such $\mathscr{F}$. This means that in the sum (II.21) one can factorize $\mathscr{T}_F(1-\mathscr{T}_{F^*})$ and the sum vanishes by (II.29). In the same way we eliminate inductively all forests containing any open quadruped, which proves Lemma II.3.

II.4. Borel Transform and Results

The coefficients in the expansion (II.1) are now precisely defined by

$$\mathbf{a}_n^N(\not{p}) = \sum_G I_G^R(\not{p}), \tag{II.31}$$

where the sum is performed over all possible sets of Wick contractions which give connected graphs G with $n(G)=n$ and $N(G)=N$, without tadpoles. The expansion (II.1) probably has a vanishing radius of convergence, and we will give a meaning only to its Borel transform in a small disk. We define the Borel transformed Schwinger functions $\mathbf{B}_N^T$ by

$$\mathbf{B}_N^T(\not{p}, t) = \delta\left(\sum_{a=1}^{N} \not{p}_a\right) \sum_{n=0}^{\infty} \frac{(-t)^n}{[n!]^2} \mathbf{a}_n^N(\not{p}). \tag{II.32}$$

Our main results are:

Theorem I. *For any graph G in euclidean Φ_4^4 theory, and any set of external momenta $\not{p}$, we have*

$$|I_G^R(\not{p})| \leqq \Gamma(\omega^R(G))(K_1|\not{p}|)^{n(G)}[f_1(G)]!, \tag{II.33}$$

where K_1 is a constant and

$$\omega^R(G) = \mathrm{Sup}\{1, N(G)/2-2\}. \tag{II.34}$$

The proof is in Sect. III and Appendices A and B.

Remark. The extra factor $f_1!$ in (II.33) does not appear in Φ_v^4 theory, $\mathrm{Re}\, v < 4$ [1]. It is characteristic of a renormalizable (and non-super-renormalizable) theory. f_1 can be seen as the maximal number of subtractions due to the renormalization. Indeed we have seen that, in the sense of (II.23) and (II.29), the Taylor operators subtract once for quadrupeds, and twice for bipeds.

Theorem II. *There exist $C(N)$ and a constant K_2 such that if $\gamma(N,n,f)$ is the number of (labeled) graphs G with $N(G)=N$, $n(G)=n$ and $f_2(G)=f$, then*

$$\gamma(N,n,f) \leqq K_2^n \frac{(n!)^2}{f!} n^{N/2} C(N). \tag{II.35}$$

The proof is in Appendix C. As a simple corollary of Theorems I, II, Lemma II.1 and formulae (II.31) and (II.32) we obtain:

Theorem III. *The expansion* (II.32) *converges and defines an analytic function of t, at least in a disk of radius* $1/|\mathcal{A}|K_1K_2$.

Remark. Inequality (II.33) is only an upper bound on I_G^R. We will come back to this point in Sect. IV, and give some examples of classes of graphs which in fact increase more slowly with $n(G)$.

III. Proof of Theorem I

III.1. Hepp's Sectors and Classification of Forests

We call $\mathfrak{S}_G$ the group of permutations of the internal lines of G. For any $\sigma\in\mathfrak{S}_G$, we call G_i^σ the subgraph of G made from the lines $\sigma(1),\ldots,\sigma(i)$. The Hepp's sector h_σ associated to σ is defined by

$$h_\sigma=\{\alpha|0\leqq\alpha_{\sigma(1)}\leqq\ldots\leqq\alpha_{\sigma(\ell)}\}. \tag{III.1}$$

We split the integral (II.19) into corresponding contributions:

$$I_G^R=\sum_{\sigma\in\mathfrak{S}_G} I_{G,\sigma}^R, \tag{III.2}$$

$$I_{G,\sigma}^R=\int_{h_\sigma}\prod_{i=1}^{\ell} d\alpha_i\left[\exp\left(-\sum_{i=1}^{\ell}\alpha_i\right)\right]Z_G^R. \tag{III.3}$$

For a given sector, we group the divergent closed forests appearing in the $\mathscr{R}$ operation according to appropriate classes. This classification is inspired by the Ω-construction of [7], which classifies the nests. The presentation here will be quite different because we want to avoid difficulties related to the unessential divergent subgraphs.

To any closed divergent forest $\mathscr{F}$ we associate a larger proper forest $\Omega^\sigma(\mathscr{F})$ in the following way: for

$$\forall F\in\mathscr{F}\cup\{G\} \text{ we consider } F\cap[G_i^\sigma\cup A_{\mathscr{F}}(F)], \tag{III.4}$$

and we take the proper components of this subgraph. By varying F in $\mathscr{F}\cup\{G\}$ and i from 1 to ℓ we thus obtain a set $\Omega^\sigma(\mathscr{F})$ of proper subgraphs.

Since

$$F=F\cap G_\ell^\sigma\cup A_{\mathscr{F}}(F),$$

$$F\subset F'\Rightarrow F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)\subsetneq F\subsetneq A_{\mathscr{F}}(F')\subsetneq F'\cap G_j^\sigma\cup A_{\mathscr{F}}(F'),\forall i,j, \tag{III.5}$$

$$F\cap F'=\emptyset\Rightarrow[F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)]\cap[F'\cap G_j^\sigma\cup A_{\mathscr{F}}(F')]=\emptyset,\forall i,j, \tag{III.6}$$

$$G_i^\sigma\subset G_j^\sigma\Rightarrow A_{\mathscr{F}}(F)\subsetneq F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)\subsetneq F\cap G_j^\sigma\cup A_{\mathscr{F}}(F)\subsetneq F, \tag{III.7}$$

it is easy to verify that $\Omega^\sigma(\mathscr{F})$ is a forest which contains $\mathscr{F}$.

Next for any $F\in\mathscr{F}$ we define an index $x(F,\mathscr{F})$, and a subgraph $X(F,\mathscr{F})$ in $\Omega^\sigma(\mathscr{F})$ by

a) $X(F,\mathscr{F})$ is a proper component of $F\cap G_{x(F,\mathscr{F})}^\sigma\cup A_{\mathscr{F}}(F)$,

b) $X^*(F,\mathscr{F})=F$,

c) $\forall i<x(F,\mathscr{F})$ and $\forall Y$ a proper component of $F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)$, $Y^*\neq F$.

We also define inductively, starting from the maximal elements of $\mathscr{F}$, an index $y(F,\mathscr{F})$ for any $F\in\mathscr{F}$, and a subforest $\mathscr{S}(\mathscr{F})$ by

d) $\mathscr{S}(\mathscr{F})=\{F|F\in\mathscr{F}\,;\, x(F,\mathscr{F})>y(F,\mathscr{F})\}$,

e) if $G\in\mathscr{F}$, $y(G,\mathscr{F})=l+1$ by convention,

f) if $F\in\mathscr{F}$, $F\neq G$, is a quadruped, $y(F,\mathscr{F})=\operatorname{Inf}_{\sigma(j)\in E} j$,

g) if $F\in\mathscr{F}$ is a biped, $y(F,\mathscr{F})=\operatorname{Sup}_{\sigma(j)\in E} j$, except when one external line of F is an external line of G, in which case we put $y(F,\mathscr{F})=\ell+1$.

In f) and g), E is the set of external lines of F, internal in $B_{\mathscr{S}(\mathscr{F})}(F)$ and $B_{\mathscr{S}(\mathscr{F})}(F)$ is defined in Sect. II as the smallest subgraph in $\mathscr{S}(\mathscr{F})$ containing F [if there is no such subgraph $B_{\mathscr{S}(\mathscr{F})}(F)=G$]. It is clear from these definitions that $G\notin\mathscr{S}(\mathscr{F})$ and that

$$x(F,\mathscr{F})\neq y(F,\mathscr{F}),\qquad \forall F\in\mathscr{F}. \tag{III.8}$$

Lemma III.1. *Let F,H be two distinct subgraphs in a closed divergent forest $\mathscr{F}$. Then if $H\notin\mathscr{S}(\mathscr{F})$ we have*

$$x(F,\mathscr{F})=x(F,\mathscr{F}-\{H\}), \tag{III.9}$$

$$y(F,\mathscr{F})=y(F,\mathscr{F}-\{H\}), \tag{III.10}$$

$$\mathscr{S}(\mathscr{F})=\mathscr{S}(\mathscr{F}-\{H\}), \tag{III.11}$$

$$\forall i,\ L(G_i^\sigma\cap F/\mathscr{F})+L(G_i^\sigma\cap H/\mathscr{F})=L(G_i^\sigma\cap F/\mathscr{F}-\{H\}). \tag{III.12}$$

Proof. (III.11) is a simple corollary of (III.9) and (III.10). (III.10) is trivial from the definitions of y and $\mathscr{S}$. (III.9) and (III.12) are trivial except when $H\in\mathscr{A}_{\mathscr{F}}(F)$. In this last case (III.12) could only be violated if the reduction by the subgraph H would add supplementary loops to $G_i^\sigma\cap F$. But this is not possible since

i) either $i>x(H,\mathscr{F})$ and $G_i^\sigma\cap H\supseteqq X(H,\mathscr{F})$,

ii) or $i\leqq x(H,\mathscr{F})<y(H,\mathscr{F})$ which means that H is disjoint from $(G_i^\sigma\cap F)-(G_i^\sigma\cap H)$ (if H is a quadruped), or joined by at most one line to it (if H is a biped).

To prove (III.9) when $H\in\mathscr{A}_{\mathscr{F}}(F)$, we remark that $F\cap G_i^\sigma\cup A_{\mathscr{F}-\{H\}}(F)$ is included in $F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)$, hence

$$x(F,\mathscr{F})\leqq x(F,\mathscr{F}-\{H\}).$$

We show the converse inequality

$$x(F,\mathscr{F})\geqq x(F,\mathscr{F}-\{H\}) \tag{III.13}$$

by distinguishing two cases:

i) either $X(F,\mathscr{F})$ is disjoint from H, then $X(F,\mathscr{F})$ is a proper component of

$$F\cap G^\sigma_{x(F,\mathscr{F})}\cup A_{\mathscr{F}-\{H\}}(F)\quad\text{and}\quad X^*(F,\mathscr{F})=F,$$

which proves (III.13),

ii) or $X(F,\mathscr{F})\supset H$, then $X(F,\mathscr{F})$ contains at least two external legs of H, internal in F, and

$$x(F,\mathscr{F})\geqq y(H,\mathscr{F})>x(H,\mathscr{F}). \tag{III.14}$$

Now $X(F,\mathscr{F})\supset H$ and $X^*(H,\mathscr{F})=H$ imply that the subgraph $Y=X(F,\mathscr{F})-[H-X(H,\mathscr{F})]$ is proper. Then $Y^*=F$ and by (III.14) Y is contained in a proper component of

$$F\cap G^\sigma_{x(F,\mathscr{F})}\cup A_{\mathscr{F}-\{H\}}(F).$$

This proves (III.13) and achieves the proof of the lemma.

Definition III.1. A closed divergent forest is called a skeleton forest if $\mathcal{S}(\mathcal{F})=\mathcal{F}$.

By simple induction one sees from Lemma III.1 that for any closed divergent forest $\mathcal{F}$, $\mathcal{S}(\mathcal{F})$ is a skeleton forest itself: $\mathcal{S}(\mathcal{S}(\mathcal{F}))=\mathcal{S}(\mathcal{F})$.

Definition III.2. For a skeleton forest $\mathcal{F}$ we define:

$$\mathcal{H}(\mathcal{F})=\left\{H\middle|\begin{array}{l}H \text{ is a closed divergent subgraph, compatible with } \mathcal{F}\\ \text{and } \mathcal{S}(\mathcal{F}\cup\{H\})=\mathcal{F}.\end{array}\right\}.$$

Lemma III.2. *$\mathcal{F}\cup\mathcal{H}(\mathcal{F})$ is a forest.*

Proof. We only have to verify that two different subgraphs H_1 and H_2 in $\mathcal{H}$ cannot overlap. But we have seen in Sect. II that the closures of two non-overlapping subgraphs cannot overlap. Thus it is sufficient to prove that $X(H_1,\mathcal{F}\cup\{H_1\})$ and $X(H_2,\mathcal{F}\cup\{H_2\})$ cannot overlap. This is true since both subgraphs belong to $\Omega^{\sigma}(\mathcal{F})$, being proper components of

$$B_{\mathcal{F}}(H_j)\cap G^{\sigma}_{x(H_j,\mathcal{F}\cup\{H_j\})}\cup A_{\mathcal{F}}(B_{\mathcal{F}}(H_j))$$

for $j=1,2$ respectively.

By Lemmas III.1 and III.2 we can conclude:

Lemma III.3. *Given a skeleton forest $\mathcal{F}$ and an arbitrary closed divergent forest $\mathcal{F}'$ we have*

$$\mathcal{S}(\mathcal{F}')=\mathcal{F}\Leftrightarrow\mathcal{F}\subseteq\mathcal{F}'\subseteq\mathcal{F}\cup\mathcal{H}(\mathcal{F}). \tag{III.15}$$

We may now use Lemma III.3 to write the $\mathcal{R}$ operation and the renormalized amplitude (III.2) in another form, by grouping all the closed divergent forests according to their skeleton forests

$$I^{R}_{G}=\sum_{\mathcal{F}}\sum_{\sigma\in\mathfrak{S}^{\mathcal{F}}_{G}} I^{\mathcal{F}}_{G,\sigma}, \tag{III.16}$$

where the first sum runs over all closed divergent forests of G, $\mathfrak{S}^{\mathcal{F}}_{G}$ is the set of $\sigma\in\mathfrak{S}_G$ such that $\mathcal{F}$ is a skeleton forest for σ and,

$$I^{\mathcal{F}}_{G,\sigma}=\int_{h_\sigma}\prod_{i=1}^{\ell}d\alpha_i\left[\exp\left(-\sum_{i=1}^{\ell}\alpha_i\right)\right]\prod_{F\in\mathcal{F}}(-\mathcal{T}_F)\prod_{H\in\mathcal{H}(\mathcal{F})}(1-\mathcal{T}_H)Z_G. \tag{III.17}$$

Although it is not obvious, the main interest of this classification lies in the fact that each integral (III.17) is absolutely convergent and may be explicitly estimated, as we will show now. Such a classification seems to realize, in a given sector h_σ, the smallest and most practical partial sums of terms in the $\mathcal{R}$ operation, which give a finite result when applied to Z_G and integrated over h_σ.

III.2. The Subtraction Process

In this paragraph we write explicitly the result of the subtractions in the integrand of (III.17). For $\mathcal{F}$ and σ fixed, $\mathcal{F}$ being a skeleton forest for σ, denote by $\mathcal{B}$ (respectively $\mathcal{Q}$) the forest of bipeds (respectively quadrupeds) in $\mathcal{F}$, with b (respectively q) elements; $\mathcal{H}$ the forest $\mathcal{H}(\mathcal{F})$; $\mathcal{B}'$ (respectively $\mathcal{Q}'$) the forest of bipeds (respectively quadrupeds) in $\mathcal{H}$, with b' (respectively q') elements; $s=|\mathcal{F}|=b+q$.

In order to apply repeatedly formulae (II.23) and (II.25)–(II.27) it is convenient to introduce the following definitions:

Definition III.3. $\mathscr{I}$ and $\mathscr{J}$ being two disjoint subforests of $\mathscr{F}\cup\mathscr{H}$, we call $S_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $T_{\mathscr{I}\bar{\mathscr{J}}}$) the set of spanning trees (respectively two-trees) of G whose restriction to F is a spanning tree of F for any $F\in\mathscr{I}$, and is not a spanning tree of F for any $F\in\mathscr{J}$. Then $U_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $V_{\mathscr{I}\bar{\mathscr{J}}}$) is defined by formula (II.4) [respectively (II.5)] in which the sum is restricted to $S_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $T_{\mathscr{I}\bar{\mathscr{J}}}$). Finally we attach a variable χ_F to every $F\in\mathscr{F}\cup\mathscr{H}$, and if $\mathscr{I}$, $\mathscr{J}$, $\mathscr{K}$ are three disjoint subforests of $\mathscr{F}\cup\mathscr{H}$, we put

$$U^{\mathscr{K}}_{\mathscr{I}\bar{\mathscr{J}}}=\sum_{\mathscr{K}'\subsetneq\mathscr{K}}\left(\prod_{F\in\mathscr{K}'}\chi_F\right)U_{\mathscr{I}\cup(\mathscr{K}-\mathscr{K}'),\,\overline{\mathscr{J}\cup\mathscr{K}'}}, \tag{III.18}$$

$$V^{\mathscr{K}}_{\mathscr{I}\bar{\mathscr{J}}}=\sum_{\mathscr{K}'\subsetneq\mathscr{K}}\left(\prod_{F\in\mathscr{K}'}\chi_F\right)V_{\mathscr{I}\cup(\mathscr{K}-\mathscr{K}'),\,\overline{\mathscr{J}\cup\mathscr{K}'}}. \tag{III.19}$$

With these notations we find:

Lemma III.4.

$$\prod_{F\in\mathscr{F}}(-\mathscr{T}_F)\prod_{H\in\mathscr{H}}(1-\mathscr{T}_H)Z_G=\sum_{\delta}Z^{\delta}_G(\not{p},\alpha), \tag{III.20}$$

where

a) *the index* δ *runs over a set of at most* $4^{f(\mathscr{F}\cup\mathscr{H})}\dfrac{[f(\mathscr{F}\cup\mathscr{H})]!}{s!}$ *elements.*

b) $$Z^{\delta}_G=\varepsilon\int_0^1\prod_{F\in\mathscr{K}}d\chi_F\prod_{F\in\mathscr{L}}(1-\chi_F)[U^{\mathscr{K}}_{\mathscr{F}}]^{-2-k}\left[\prod_{m=1}^{k}P_m\right]\exp[-W(\chi,\alpha,\not{p})], \tag{III.21}$$

(ε, $\mathscr{K}$, $\mathscr{L}$, k, P_m, and W depend on the index δ).

c) $$\varepsilon=\pm1;\quad W=V^{\mathscr{K}}_{\mathscr{F}}/U^{\mathscr{K}}_{\mathscr{F}}\geqq0;\quad P_m\geqq0, \tag{III.22}$$

$$\mathscr{L}\subseteq\mathscr{K}\subseteq\mathscr{H}, \tag{III.23}$$

$$k\leqq2f(\mathscr{F}\cup\mathscr{H}). \tag{III.24}$$

d) $$P_m=U^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}\ \text{or}\ V^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}\ \text{(in both cases denote it}\ P^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}). \tag{III.25}$$

Moreover $P_m=V^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}$ *for at most* $f(\mathscr{F}\cup\mathscr{H})$ *values of* m.

e)
$$\left.\begin{aligned}
F\in\mathscr{Q}&\Rightarrow F\in\mathscr{I}_m\forall m,\\
F\in\mathscr{Q}'&\Rightarrow\exists!m,\quad F\in\mathscr{J}_m,\\
F\in\mathscr{B}&\Rightarrow\text{ either } F\in\mathscr{I}_m\forall m,\text{ or }\exists!m,\quad F\in\mathscr{J}_m,\\
F\in\mathscr{B}'&\Rightarrow\exists m_1,m_2,\quad F\in\mathscr{J}_m\Leftrightarrow m=m_1\text{ or }m=m_2.
\end{aligned}\right\} \tag{III.26}$$

f) $$\forall m,\ \mathscr{F}\subseteq\mathscr{I}_m\cup\mathscr{J}_m\quad\text{and}\quad\mathscr{F}\cup\mathscr{K}\subseteq\mathscr{I}_m\cup\mathscr{J}_m\cup\mathscr{K}_m. \tag{III.27}$$

g) *If* $F\in\mathscr{J}_m$ *and* $F'\in\mathscr{A}_{\mathscr{F}\cup\mathscr{K}}(F)$, *then* $F'\in\mathscr{I}_m\cup\mathscr{J}_m$, *except perhaps if* $F\in\mathscr{B}'$ *and* $F'\in\mathscr{B}\cup\mathscr{Q}'$; *in this last case,* m_1 *and* m_2 *being the indices defined in* (III.26), $F\in\mathscr{J}_{m_1}\Rightarrow F'\in\mathscr{I}_{m_2}\cup\mathscr{J}_{m_2}$ *and* $F\in\mathscr{J}_{m_2}\Rightarrow F'\in\mathscr{I}_{m_1}\cup\mathscr{J}_{m_1}$.

Proof. The proof is by induction on b, q, b', q'. We add new subgraphs one by one, starting from the maximal elements of the final forest. Assuming Lemma III.4 at a given step, we add a new subgraph F and perform the corresponding $\mathscr{T}_F$ or $1-\mathscr{T}_F$ operation. The dependence on the corresponding χ_F is obtained by replacing $\mathscr{K}$ by $\mathscr{K}\cup\{F\}$. Then we use (II.23), (II.25), (II.26) or (II.27) and verify that Lemma III.4 remains true. Among all possible cases, we write explicitly the most complicated one, i.e.: $F\in\mathscr{B}'$ (hence we perform $1-\mathscr{T}_F$) and $B_{\mathscr{F}}(F)\in\mathscr{B}'$ [hence $B_{\mathscr{F}}(F)\in\mathscr{J}_{m_1}$ and $B_{\mathscr{F}}(F)\in\mathscr{J}_{m_2}$, which is the most difficult case for the verification of item g)]. We simplify the notations by numbering the indices in such a way that $m_1=1$, $m_2=2$. Then the $1-\mathscr{T}_F$ operator subtracts 0, 1, or 2 times and we obtain

$$(1-\mathscr{T}_F)Z_G^\delta = Z_0 + Z_1 + Z_2, \tag{III.28}$$

$$Z_0 = \varepsilon \int_0^1 \prod_{F'\in\mathscr{K}} d\chi_{F'} \prod_{F'\in\mathscr{L}} (1-\chi_{F'}) [U_{\mathscr{F}}^{\mathscr{K}}]^{-2-k} \exp\left(-\frac{V_{\mathscr{F}}^{\mathscr{K}}}{U_{\mathscr{F}}^{\mathscr{K}}}\right)$$
$$\cdot \sum_{1\le t<t'\le k} \left[\prod_{\substack{m=1\\ m\neq t}}^{t'-1} P^{\mathscr{K}}_{\mathscr{J}_m\cup\{F\},\,\overline{\mathscr{J}_m}}\right]\left[\prod_{m=t'+1}^{k} P^{\mathscr{K}}_{\mathscr{J}_m,\,\overline{\mathscr{J}_m}}\right] P^{\mathscr{K}}_{\mathscr{J}_t,\,\overline{\mathscr{J}_t\cup\{F\}}} P^{\mathscr{K}}_{\mathscr{J}_{t'},\,\overline{\mathscr{J}_{t'}\cup\{F\}}}, \tag{III.29}$$

$$Z_1 = \varepsilon \int_0^1 \prod_{F'\in\mathscr{K}\cup\{F\}} d\chi_{F'} \prod_{F'\in\mathscr{L}} (1-\chi_{F'}) \exp\left(-\frac{V_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}}{U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}}\right) \sum_{t=1}^{k} \prod_{\substack{m=1\\ m\neq t}}^{k} P^{\mathscr{K}}_{\mathscr{J}_m\cup\{F\},\,\overline{\mathscr{J}_m}}$$
$$\cdot P^{\mathscr{K}}_{\mathscr{J}_t,\,\overline{\mathscr{J}_t\cup\{F\}}} \left\{-\frac{(2+k)U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}}{[U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}]^{2+k+1}} + \frac{U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}} V^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} - U^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} V^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}}{[U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}]^{2+k+2}}\right\}, \tag{III.30}$$

$$Z_2 = \varepsilon \int_0^1 \prod_{F'\in\mathscr{K}\cup\{F\}} d\chi_{F'} \prod_{F'\in\mathscr{L}\cup\{F\}} (1-\chi_{F'}) \exp\left(-\frac{V_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}}{U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}}\right) \prod_{m=1}^{k} P^{\mathscr{K}}_{\mathscr{J}_m\cup\{F\},\,\bar{\mathscr{J}}_m}$$
$$\cdot\left\{(2+k)(3+k)\frac{[U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}]^2}{[U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}]^{2+k+2}}\right.$$
$$-2(2+k+1)U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}\frac{U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}} V^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} - U^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} V^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}}{[U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}]^{2+k+3}}$$
$$\left.+\frac{[U^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}} V^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} - U^{\mathscr{K}}_{\mathscr{F}\cup\{F\}} V^{\mathscr{K}}_{\mathscr{F},\overline{\{F\}}}]^2}{[U_{\mathscr{F}}^{\mathscr{K}\cup\{F\}}]^{2+k+4}}\right\}. \tag{III.31}$$

We can verify Lemma III.4 on these formulae. The other cases, following F and $B_{\mathscr{F}}(F)$ belong to $\mathscr{B}$, $\mathscr{Q}$, $\mathscr{B}'$, $\mathscr{Q}'$, give formulae which are shorter. The verification of items a)–g) is easier, and is left to the reader.

Corollaries of Lemma III.4. *From* (III.21) *and* (III.22) *we conclude*

$$|Z_G^\delta(\rho,\alpha)| \le Y_G^\delta(\rho,\alpha) = \int_0^1 \prod_{F\in\mathscr{K}} d\chi_F \left[\prod_{m=1}^{k} P_m\right] [U_{\mathscr{F}}^{\mathscr{K}}]^{-2-k}. \tag{III.32}$$

From g) and (III.26), it is easy to verify

$$\forall\delta,\ \sum_{m=1}^{k} \sum_{F\in\mathscr{J}_m} \ell(F/\mathscr{J}_m\cup\mathscr{J}_m) \le 2 \sum_{F\in\mathscr{F}\cup\mathscr{K}} \ell(F/\mathscr{F}\cup\mathscr{K}) \le 2\ell(G). \tag{III.33}$$

III.3. Change of Variables and Estimates

In the integrand of (III.17) expressed by means of Lemma III.4, we perform the usual change of variables

$$\alpha_k = \prod_{G_i^\sigma \ni k} \beta_i . \tag{III.34}$$

Each variable β_i varies from 0 to 1, except β_ℓ which varies from 0 to ∞. Since the various polynomials U, V, P_m are homogeneous with respect to β_ℓ, we perform first the β_ℓ integration, [using (III.26) if G is divergent] and we obtain from (III.17), (III.20), and (III.32)

$$|I^{\mathscr{F}}_{G,\sigma}| \leqq \int_0^1 \dots \int_0^1 \prod_{i=1}^{\ell-1} \beta_i^{i-1} d\beta_i \Big[\sum_\delta Y_G^\delta(\not{p}, \beta_i) \Big] \cdot \Gamma(\omega^R(G)) . \tag{III.35}$$

Since the global homogeneity in β_ℓ has been taken out, $Y_G^\delta(\not{p}, \beta_i)$ as defined in (III.32) is only a function of $\beta_1, \dots, \beta_{\ell-1}$, and $\omega^R(G)$ is defined in (II.34).

Lemma III.5.

$$\forall \delta, \int_0^1 \dots \int_0^1 \prod_{i=1}^{\ell-1} \beta_i^{i-1} d\beta_i Y_G^\delta(\not{p}, \beta_i) \leqq (K_3 |\not{p}|)^{n(G)} \prod_{i=1}^{\ell-1} \frac{1}{\tilde{N}_i(\sigma)} , \tag{III.36}$$

where

$$\tilde{N}_i(\sigma) = \sum_{\substack{F \in \mathscr{F} \cup \{G\} \\ G_i^\sigma \cap F / \mathscr{F} \neq F / \mathscr{F}}} N(G_i^\sigma \cap F / \mathscr{F}) . \tag{III.37}$$

The proof of Lemma III.5 is rahter lengthy and we give it in the next subsection. As a consequence of the definition of the skeleton forests, $\tilde{N}_i(\sigma)$ never vanish. Then we are left with the sum over Hepp's sectors. This problem is treated in Appendix B, where the following result is proved:

Lemma III.6.

$$\forall \mathscr{F}, \sum_{\sigma \in \mathfrak{S}_G^{\mathscr{F}}} \frac{1}{S!} \prod_{i=1}^{\ell-1} \frac{1}{\tilde{N}_i(\sigma)} \leqq K_4^{n(G)} . \tag{III.38}$$

From Lemma C.1 and Definition II.6, the index δ runs over a set of at most $4^n f_1 !/s!$ elements. Finally we can bound the sum in (III.16), using Appendix A to bound the possible choices of $\mathscr{F}$, and (III.35), (III.36), (III.38) to bound $\sum_{\sigma \in \mathfrak{S}_G^{\mathscr{F}}} I^{\mathscr{F}}_{G,\sigma}$. Hence Theorem I is proved.

III.4. Proof of Lemma III.5

To prove Lemma III.5 we will define a natural correspondence between the subsets $S_{\mathscr{J}_m \bar{\mathscr{J}}_m}$ or $T_{\mathscr{J}_m \bar{\mathscr{J}}_m}$ which index the sums P_m in the numerator of (III.32), and the subset $S_{\mathscr{F}}$ which indexes the denominator $U^{\mathscr{K}}_{\mathscr{F}}$ of (III.32).

To fix the language we only consider the case $P_m = U^{\mathscr{K}_m}_{\mathscr{J}_m \bar{\mathscr{J}}_m}$. The case $P_m = V^{\mathscr{K}_m}_{\mathscr{J}_m \bar{\mathscr{J}}_m}$ would be treated along the same lines, with minor changes which we

indicate later in a remark. In this case, we bound at the beginning the dependence on the external momenta by (II.7).

Let us now describe a first elementary mapping φ_F in the set of spanning trees. We put $\mathscr{I}_m \cup \mathscr{J}_m = \mathscr{E}_m$, and $\forall F \in \mathscr{J}_m$ we define $\mathscr{E}_{mF} = \{F' | F' \in \mathscr{E}_m ; F' \subset F\}$. Let S be a spanning tree of G whose restriction to each $F' \in \mathscr{E}_{mF}$ is a spanning tree of F'. Then we define a mapping φ_F from $S_{\mathscr{E}_{mF}}$ into $S_{\mathscr{E}_{mF} \cup \{F\}}$ in the following way:

1) If S is a one-tree in $F/\mathscr{E}_m$ (hence a one-tree in F) we put

$$\varphi_F(S) = S . \tag{III.39}$$

2) If S is not a one-tree in $F/\mathscr{E}_m$, by Lemma II.2 S is not a one-tree in $X_F = X(F, \mathscr{E}_m)/\mathscr{E}_m$; hence $S \cap X_F$ has several connected components (some of them eventually reduced to a single vertex); since each of these connected components must contain an external vertex of F, their number $c_F(S)$ satisfies:

$$2 \leqq c_F(S) \leqq N(F) \leqq 4 . \tag{III.40}$$

We choose an arbitrary connected component γ of $S \cap X_F$. Since X_F is connected, and all its vertices belong to S, there exists another connected component γ' of $S \cap X_F$ linked to γ by one line $\lambda' \notin S$. Then if i' is the rank of λ' in the sector, $\lambda' = \sigma(i')$, we have by (III.9) and (III.27):

$$i' \leqq x(F, \mathscr{E}_m) = x(F, \mathscr{F}) . \tag{III.41}$$

Next we define an external line λ of F by distinguishing three cases:

a) If $B_{\mathscr{F}}(F) \in \mathscr{Q}$, then S is a one-tree in $B_{\mathscr{F}}(F)$ by (III.26). Then there is a path in $S \cap B_{\mathscr{F}}(F)$ from γ to γ', which contains at least two external lines of F, one attached to γ, the other one to γ'. Among these two lines, we call λ the one which has the highest rank in the sector σ.

b) If $B_{\mathscr{F}}(F) \in \mathscr{B}$ and F is a biped, there is a path in S from γ to γ' which contains both external lines of F, one attached to γ and the other one to γ'. Moreover these two lines are internal in $B_{\mathscr{F}}(F)$ since $B_{\mathscr{F}}(F)$ is proper. We define λ in the same way as in case a).

c) If $B_{\mathscr{F}}(F) \in \mathscr{B}$ and F is a quadruped, there is a path in S from γ to γ' which contains at least two external lines of F, one attached to γ, the other one to γ'. These two lines cannot be both external lines of $B_{\mathscr{F}}(F)$ since F is closed, and we define λ as one of these two lines which is internal in $B_{\mathscr{F}}(F)$ (any one of them if both are).

In the three cases a)–c), we conclude that if $\lambda = \sigma(i)$ then $i \geqq y(F, \mathscr{E}_m)$, hence by (III.27) and (III.10)

$$\lambda = \sigma(i) \Rightarrow i \geqq y(F, \mathscr{F}) . \tag{III.42}$$

By construction $S \cup \{\lambda'\} - \{\lambda\}$ is a one-tree in G and has $c_F(S) - 1$ connected components in $F/\mathscr{E}_m$. Iterating this operation $c_F(S) - 1$ times, we build

$$\varphi_F(S) = S \cup \{\lambda'_1, \ldots, \lambda'_{c_F(S)-1}\} - \{\lambda_1, \ldots, \lambda_{c_F(S)-1}\} , \tag{III.43}$$

which is a one-tree in $F/\mathscr{E}_m$, hence in F.

We note that by this construction, there are at most $2^4 \cdot 2^{\ell(F/\mathscr{E}_m)}$ one-trees S such that $\varphi_F(S)$ is equal to some given one-tree.

Remark. When $P_m = V^{\mathscr{K}_m}_{\mathscr{I}_m \bar{\mathscr{I}}_m}$, we associate a one-tree $\varphi_F(T)$ to a two-tree T by an almost identical algorithm: if γ and γ' cannot be joined by a path in T, λ cannot be defined and we simply add λ' without removing any line, which gives a one-tree. If λ can be defined at each step, we simply add an arbitrary line at the end, to get a one-tree.

Now we define a second mapping φ_m for any $S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m}$ by

$$\varphi_m(S) = \left[\prod_{F \in \mathscr{I}_m} \varphi_F\right](S), \tag{III.44}$$

the product (in the sense of the composition of mappings) being taken in an order compatible with the inclusion relations of the subgraphs F: if $\mathscr{D}_{mF}$ is the set of $F' \in \mathscr{E}_m$ such that $\varphi_{F'}$ is applied before φ_F, we have $F' \in \mathscr{D}_{mF} \Rightarrow F \subsetneq F'$. By this definition, $\varphi_m(S)$ is a one-tree; $\forall F'$ compatible with $\mathscr{E}_m$,

$$S \in S_{\mathscr{E}_{mF} \cup \{F'\}} \Rightarrow \forall F \in \mathscr{E}_m, \varphi_F(S) \quad \text{is a one-tree in } F', \tag{III.45}$$

since the number of lines of $S \cap F'$ can only increase by applying φ_F. The mapping φ_m maps $S_{\mathscr{I}_m, \bar{\mathscr{I}}_m}$ into $S_{\mathscr{E}_m} \subseteq S_{\mathscr{F}}$ since $\mathscr{F} \subseteq \mathscr{E}_m$ by (III.27). By construction, the number of one-trees which become a given one-tree S_1 by this mapping is bounded by

$$|\{S | \varphi_m(S) = S_1\}| \leqq 2^{4|\mathscr{I}_m|} \cdot 2^{\sum_{F \in \mathscr{I}_m} \ell(F/\mathscr{E}_m)}. \tag{III.46}$$

By (III.18) we have

$$P_m = \sum_{S \in S_{\mathscr{I}_m, \bar{\mathscr{I}}_m}} \left[\prod_{\substack{F \in \mathscr{K}_m \\ S(F) \neq S}} \chi_F\right] \cdot u(S), \tag{III.47}$$

$$U^{\mathscr{K}}_{\mathscr{F}} = \sum_{S \in S_{\mathscr{F}}} \left[\prod_{\substack{F \in \mathscr{K} \\ S(F) \neq S}} \chi_F\right] \cdot u(S), \tag{III.48}$$

where

$$u(S) = \prod_{i' \notin S} \prod_{G \ni i'} \beta_i. \tag{III.49}$$

By (III.27) $\mathscr{K} \subseteq \mathscr{K}_m \cup \mathscr{E}_m$. Therefore by (III.45) and (III.46)

$$\frac{P_m}{U^{\mathscr{K}}_{\mathscr{F}}} \leqq 2^{4|\mathscr{I}_m|} \cdot 2^{\sum_{F \ni \mathscr{I}_m} \ell(F/\mathscr{E}_m)} \cdot \operatorname*{Sup}_{\substack{S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m} \\ S' = \varphi_m(S)}} \frac{u(S)}{u(S')}. \tag{III.50}$$

Lemma III.7.

$$\operatorname*{Sup}_{\substack{S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m} \\ S' = \varphi_m(S)}} \frac{u(S)}{u(S')} \leqq \frac{\prod_{F \in \mathscr{I}_m \cap \mathscr{H}} \prod_{i \in M_F} \beta_i}{\prod_{F \in \mathscr{I}_m \cap \mathscr{B}} \prod_{i \in M_F} \beta_i}, \tag{III.51}$$

where

$$M_F = \{i | x(F, \mathscr{F}) \leqq i < y(F, \mathscr{F}) \quad \text{if} \quad F \in \mathscr{H}; y(F, \mathscr{F}) \leqq i < x(F, \mathscr{F}) \text{ if } F \in \mathscr{F}\}. \tag{III.52}$$

Proof. We analyse the action of φ_m by looking at the composition of mappings: for $S \in S_{\mathcal{J}_m \mathcal{J}_m}$ and $F \in \mathcal{J}_m$ we examine how φ_F acts on $S' = \left[\prod_{F' \in \mathcal{Q}_{mF}} \varphi_{F'}\right](S)$.

a) If $F \in \mathcal{B}$, then by (III.40) φ_F acts on S' by changing at most one line λ of S' into another line λ', and by (III.41) and (III.42) this implies

$$\frac{u(\varphi_F(S'))}{u(S')} \leqq \left[\prod_{y(F,\mathcal{F}) \leqq i < x(F,\mathcal{F})} \beta_i\right]^{-1}. \tag{III.53}$$

b) If $F \in \mathcal{H}$ and S' is not a one-tree in F, then $\varphi_F(S')$ is given by (III.43), and we have directly from (III.41) and (III.42)

$$\frac{u(\varphi_F(S'))}{u(S')} \leqq \prod_{x(F,\mathcal{F}) \leqq i < y(F,\mathcal{F})} \beta_i. \tag{III.54}$$

c) If $F \in \mathcal{H}$ and S' is a one-tree in F, then $\varphi_F(S') = S'$.

Since $S \in S_{\mathcal{J}_m \mathcal{J}_m}$, S is not a one-tree in F, and there must exist some $F_1 \subset F$ with

$$\begin{cases} S_1 = \left[\prod_{F' \in \mathcal{Q}_{mF_1}} \varphi_{F'}\right](S) & \text{not a one-tree in } F, \\ \varphi_{F_1}(S_1) & \text{one tree in } F. \end{cases}$$

In other words, φ_{F_1} removes from S_1 at least one line λ which is external in F and F_1, and internal in $B_{\mathcal{F}}(F_1)$. This implies

$$F_1 \subset F \subset B_{\mathcal{F}}(F_1).$$

More generally the whole set $\{F_1, F_2, \ldots, F_r\}$ of subgraphs F_j, such that $F_1 \subsetneq F_j \subset B_{\mathcal{F}}(F_1)$ and λ is external in F_j, is totally ordered by inclusion: $F_1 \subset F_2 \subset \ldots \subset F_r \subset B_{\mathcal{F}}(F_1)$. Applying inductively the argument given in the proof of Lemma III.1, we have as a generalization of (III.14)

$$x(F_1, \mathcal{F}) < y(F_1, \mathcal{F}) \leqq x(F_2, \mathcal{F}) < y(F_2, \mathcal{F}) \leqq \ldots \leqq x(F_r, \mathcal{F}) < y(F_r, \mathcal{F}) \leqq k,$$

where k is the rank of λ in the sector: $\lambda = \sigma(k)$. From (III.41) and (III.42) again, we have

$$\frac{u(\varphi_{F_1}(S_1))}{u(S_1)} \leqq \prod_{x(F_1,\mathcal{F}) \leqq i < k} \beta_i \leqq \prod_{j=1}^{r} \prod_{x(F_j,\mathcal{F}) \leqq i < y(F_j,\mathcal{F})} \beta_i. \tag{III.55}$$

In particular, like (III.54), (III.55) contains the factor

$$\prod_{x(F,\mathcal{F}) \leqq i < y(F,\mathcal{F})} \beta_i.$$

Gathering the results a)–c) with the definition (III.44) achieves the proof of Lemma III.7.

By Lemma III.7, (III.25), (III.26), (III.32), (III.33), and (III.50), we can perform trivially the χ_F integrations and use $U^{\mathcal{H}}_{\mathcal{F}}(\beta_i, \chi_F) \geqq U_{F \cup \mathcal{H}}(\beta_i)$ to get

$$Y^{\delta}_G(\not{p}, \beta_i) \leqq |\not{p}|^{n(G)} \frac{2^{2\ell(G)} \cdot 2^{4n(G)}}{[U_{\mathcal{F} \cup \mathcal{H}}(\beta_i)]^2} \cdot \frac{\left[\prod_{F \in \mathcal{B}'} \prod_{i \in M_F} \beta_i\right]\left[\prod_{F \in \mathcal{B}'} \prod_{i \in M_F} \beta_i\right]^2}{\prod_{F \in \mathcal{B}} \prod_{i \in M_F} \beta_i} \tag{III.56}$$

[we have used (III.26) to obtain: $\sum_{m=1}^{k} |\mathscr{J}_m| \leqq f(\mathscr{H}) + b(\mathscr{F}) \leqq f_1(G) \leqq n(G)$]. From (III.56) we deduce

$$Y_G^{\delta}(\not{p}, \beta_i) \leqq |\not{p}|^{n(G)} \cdot 2^{2\ell(G)} \cdot 2^{4n(G)} \prod_{i=1}^{\ell-1} \beta_i^{\eta_i}, \tag{III.57}$$

with

$$\eta_i = -2 \sum_{F \in \mathscr{F} \cup \mathscr{H} \cup \{G\}} L(G_i^{\sigma} \cap F / \mathscr{F} \cup \mathscr{H}) + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{L}' \\ M_F \ni i}} 1 - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1 . \tag{III.58}$$

Therefore (III.36) and Lemma III.5 will be proved by the following:

Lemma III.8.

$$\forall i, \quad i + \eta_i \geqq \frac{\tilde{N}_i(\sigma)}{6}. \tag{III.59}$$

Proof. Using inductively (III.12) we can change (III.58) into

$$i + \eta_i = \sum_{F \in \mathscr{F} \cup \{G\}} \omega(G_i^{\sigma} \cap F / \mathscr{F}) + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{L}' \\ M_F \ni i}} 1 - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1 . \tag{III.60}$$

In this formula, any connected component C of $G_i^{\sigma} \cap F / \mathscr{F}$ can contain vertices with only two lines, by reduction of elements of $\mathscr{B}$. Hence if $b(C)$ is the number of such vertices, (II.11) has to be modified into

$$\omega(C) = \frac{N(C)}{2} - 2 + b(C). \tag{III.61}$$

But for $F \in \mathscr{B}$, $i \in M_F$, we have $y(F, \mathscr{F}) \leqq i < x(F, \mathscr{F})$; therefore by definition of $y(F, \mathscr{F})$ there exists in $B_{\mathscr{F}}(F)/\mathscr{F}$ a connected component C of $G_i^{\sigma} \cap B_{\mathscr{F}}(F)/\mathscr{F}$ which contains the reduction vertex of F.

Thus we have

$$\sum_{F \in \mathscr{F}} \left[\sum_{\substack{C \text{ conn. comp.} \\ \text{of } G_i^{\sigma} \cap B_{\mathscr{F}}(F)/\mathscr{F}}} b(C) \right] - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1 \geqq \sum_{\substack{F \in \mathscr{B} \\ x(F, \mathscr{F}) \leqq i}} 1 , \tag{III.62}$$

and (III.60) becomes

$$i + \eta_i \geqq \sum_{F \in \mathscr{F} \cup \{G\}} \left[\sum_{\substack{C \text{ conn. comp.} \\ \text{of } G_i^{\sigma} \cap F / \mathscr{F}}} \left(\frac{N(C)}{2} - 2 \right) \right] + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{L}' \\ M_F \ni i}} 1 + \sum_{\substack{F \in \mathscr{B} \\ x(F, \mathscr{F}) \leqq i}} 1 . \tag{III.63}$$

We can now evaluate $i + \eta_i$, by distinguishing, given $F \in \mathscr{F} \cup \{G\}$, the possible cases for connected components C of $G_i^{\sigma} \cap F / \mathscr{F}$:

a) If $\frac{N(C)}{2} - 2 > 0$, then $\frac{N(C)}{2} - 2 \geqq \frac{N(C)}{6}$. (III.64)

b) If $\frac{N(C)}{2} - 2 \leqq 0$ and C is not proper, then C is represented by Figs 2 or 3 and contains a proper biped $B(C)$ which satisfies $B(C) = B'/\mathscr{F}$, $B' \in \mathscr{B}'$ and $M_{B'} \ni i$ from

the definitions of Subsect. III.1. Hence we have

$$\left(\frac{N(C)}{2}-2\right)+2=\frac{N(C)}{2}>\frac{N(C)}{6}. \tag{III.65}$$

c) If $\frac{N(C)}{2}-2\leqq 0$ and C is proper but not closed, then C is a quadruped, $C^*=B'/\mathscr{F}$ and $B'\in\mathscr{B}\cup\mathscr{B}'$.

Either $B'\in\mathscr{B}'$ and $M_{B'}\ni i$. Hence

$$\left(\frac{N(C)}{2}-2\right)+2=2>\frac{N(C)}{6}. \tag{III.66}$$

Or else $B'\in\mathscr{B}$; then $X(B',\mathscr{F})/\mathscr{F}\subset C$, that is $x(B',\mathscr{F})\leqq i$.
Hence

$$\left(\frac{N(C)}{2}-2\right)+1=1>\frac{N(C)}{6}. \tag{III.67}$$

d) If $\frac{N(C)}{2}-2\leqq 0$ and C is proper and closed, then $C=F'/\mathscr{F}$ with the following possibilities:

$$F'\in\mathscr{L}',\quad M_{F'}\ni i,\quad \text{hence}\quad \left(\frac{N(C)}{2}-2\right)+1=1>\frac{N(C)}{6}, \tag{III.68}$$

$$F'\in\mathscr{B}',\quad M_{F'}\ni i,\quad \text{hence}\quad \left(\frac{N(C)}{2}-2\right)+2=1>\frac{N(C)}{6}, \tag{III.69}$$

$$F'\in\mathscr{L},\quad \text{hence}\quad F'=F\quad \text{and}\quad \frac{N(C)}{2}-2=0, \tag{III.70}$$

$$F'\in\mathscr{B},\quad \text{hence}\quad F'=F, x(F,\mathscr{F})\leqq i\quad \text{and}\quad \left(\frac{N(C)}{2}-2\right)+1=0. \tag{III.71}$$

All these cases are mutually exclusive and allow us to bound the sum in the right-hand side of (III.63), proving Lemma III.8. The cases (III.70) and (III.71) correspond to the restriction $G_i^{\sigma}\cap F/\mathscr{F}\neq F/\mathscr{F}$ in (III.37).

IV. Discussion

IV.1. Discussion of Theorem 1

We mainly want to emphasize that Theorem I gives only an upper bound on the renormalized Feynman amplitudes, but that their true behaviour may be quite different. First we note that (III.33) does not describe the true behaviour of these amplitudes at large momenta, because we used in (III.32) the very crude bound $\exp(-W)\leqq 1$. The asymptotic behaviour of a given amplitude, when any set of external momenta is scaled to infinity, can be computed in a systematic way, for instance by the multiple or complete Mellin techniques [8]. The result is an expansion in powers and powers of logarithms, which generalizes the Weinberg

theorem. In particular the leading term, when all external momenta are scaled by ϱ and ϱ goes to infinity, behaves like

$$\varrho^{\underset{F\subsetneq G}{\mathrm{Sup}}(-2\omega(F))} \ln^{\zeta(G)}\varrho\,, \tag{IV.1}$$

where

$$\zeta(G)=\underset{\mathcal{N}}{\mathrm{Sup}}|\mathcal{N}|-\delta\,,\quad \delta=0 \quad \text{or} \quad 1\,,$$

$$\mathcal{N}=\left\{F_1\subset\ldots\subset F_r\ldots\subset F_{|\mathcal{N}|},\ \text{with}\quad \omega(F_r)=\underset{F\subsetneq G}{\mathrm{Inf}}\,(\omega(F))\forall r\right\}. \tag{IV.2}$$

We believe that it is possible to take into account the decrease of $\exp(-W)$ in (III.21), to improve Theorem I and obtain bounds at large order with the right behaviour (IV.1) at large momenta. More generally it would be interesting to estimate jointly the large order and the various large momenta behaviours of Feynman amplitudes. Both problems are linked, in fact, the large momenta behaviour of the chain in Fig. 4 led to the discovery of the factorial behaviour of some Feynman amplitudes [4].

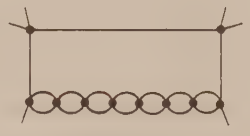

Fig. 4

Even at fixed external momenta, Theorem I does not give the right large order behaviour (up to exponentials of the order). As a first trivial example, when the graph is one-vertex reducible, one can bound its amplitude by the product of the bounds on its one-vertex irreducible components, which generally improves the bound of Theorem I.

We also remark that a biped does not necessarily have to be counted twice in the true factorial behaviour at large order. As a simple example of this fact, one can study the graph of Fig. 5 and prove easily:

Fig. 5

Proposition IV.1. *Let G_m be the graph of Fig. 5, with n vertices and $m=n/2-3/2$ proper bipeds. Then there exists two positive constants K_6 and K_7 such that*

$$K_6^n m!\leqq I^R_{G_m}(0)\leqq K_7^n m! \tag{IV.3}$$

Therefore the large order behaviour of $I^R_{G_m}$ is given by $(n!)^{1/2}$ rather than $n!$. This discrepancy could perhaps be corrected by identifying all the reduction lines in a chain of proper bipeds. In the properly modified α-parametric representation,

there would be fewer Hepp's sectors; our definition of skeleton forests in Subsect. III.1 would become more natural by treating bipeds and quadrupeds in the same way.

Finally we must mention that we did not try to find the smallest possible constants in Theorems I and II. By looking at many places in Sect. III and in the appendices, one could lower these constants.

IV.2. Analytic Continuation in the Borel Plane and Renormalons

Although the expansion in the coupling constant is Borel summable for Φ_2^4 or Φ_3^4 field theory [2], this is not believed to be the case for Φ_4^4. The analytic continuation of the function $\mathbf{B}_N^T$ depends in a crucial way on the signs of the coefficients $\mathbf{a}_n^N$, and the renormalization is supposed to destroy the alternation of signs in the perturbative expansion. More precisely it has been argued [5] that the partial series retaining only the graphs of Fig. 4 (since they behave like $K^n n!$ and present no alternation of signs) is responsible for the appearance of a singularity ("renormalon") of $\mathbf{B}_N^T(t)$, at the positive real value $t = 1/K$. Of course this is not a rigorous argument: there could be a conspiracy between the graphs of Fig. 4 and the other ones, which cancels the singularity. Such a cancellation would certainly be spectacular, but we are not sure it can be ruled out a priori. For example, if we separate the expansion of the Schwinger functions of Φ_2^4 into two parts, respectively given by the amplitudes with even and odd number of vertices, each of these parts presents singularities on the positive real axis of the Borel plane. Constructive field theory proves that these singularities disappear when we add them, a property which is not obvious at all at the perturbative level.

On this open problem of the existence of renormalons, we may note that our study generalizes the argument based on the graphs of Fig. 4. For example the amplitudes of the G_m in Fig. 5 also define a partial series with no alternation of signs. Since their number is about $n!$, and they grow as $(n!)^{1/2}$, the corresponding partial series in the Borel plane has an infinite radius of convergence. However, it defines a function which incrases like $\exp(t^2)$ at infinity on the real positive axis, destroying Borel summability. Such a feature can be thought of as a "renormalon at infinity". Since some factorial behaviour is a very common property of the renormalized amplitudes, there could be many such "renormalons at infinity". Again, we cannot say whether they cancel or not in the analytic continuation of the whole function $\mathbf{B}_N^T$.

Finally let us mention that our methods can be applied to other renormalizable theories, in spite of the tedious complications due to spinor fields. Our opinion is that the Borel transform exists locally again, for the Yukawa model or quantum electrodynamics in 4 dimensions for instance. Yet the problem of the analytic continuation may present very different features, depending on the model.

Appendix A

In this appendix we bound the number of closed divergent forests in the graphs with n vertices.

A.1. Overlapping Divergent Subgraphs

If two subgraphs F_1 and F_2 overlap, we have evidently

$$N(F_1)+N(F_2)=N(F_1\cup F_2)+N(F_1\cap F_2), \tag{A.1}$$

where $N(F)$ is the number of external lines of F. If F_1 and F_2 are divergent, $N(F_1)$ and $N(F_2)$ may be equal to 2 or 4. Moreover if F_1 and F_2 are proper, $F_1\cap F_2$ must be linked to $F_1-(F_1\cap F_2)$ by at least two lines, and to $F_2-(F_1\cap F_2)$ by at least two different lines. Thus

$$N(F_1\cap F_2)\geqq 4. \tag{A.2}$$

Hence two proper bipeds cannot overlap if there is no vacuum graph, and we are left with the following cases:

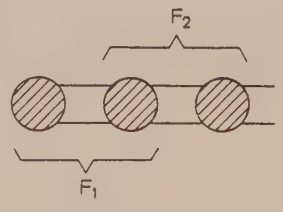

Fig. 6

$N(F_1)=2;\quad N(F_2)=4;\quad N(F_1\cup F_2)=2;\quad N(F_1\cap F_2)=4$ (Fig. 6)

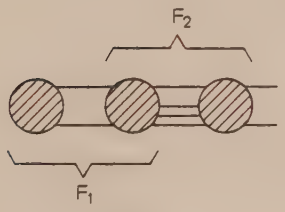

Fig. 7

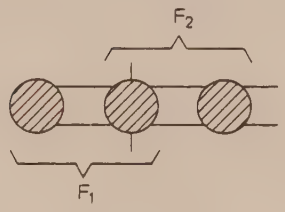

Fig. 8

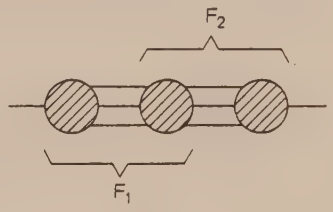

Fig. 9

$N(F_1)=N(F_2)=4;\quad N(F_1\cup F_2)=2;\quad N(F_1\cap F_2)=6$ (Figs. 7–9)

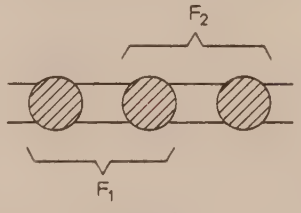

Fig. 10

$N(F_1)=N(F_2)=N(F_1\cup F_2)=N(F_1\cap F_2)=4$. (Fig. 10)

If F_1 and F_2 are closed, Figs. 6–8 are forbidden. We see in particular that a proper biped cannot overlap with any closed divergent subgraph.

A.2. Closed Divergent Forests

A closed divergent subgraph F of G is said to be maximal if $F\subset G$, and there exists no other closed divergent subgraph F' satisfying $F\subset F'\subset G$. A closed divergent

forest is maximal if there is no other closed divergent forest containing it. Given a maximal closed divergent forest $\mathcal{F}$:

i) either the biggest subgraphs in $\mathcal{F}$ (apart from G itself if $G \in \mathcal{F}$) are the maximal closed divergent subgraphs of G.

ii) or there exists a maximal subgraph of G, say F_1, with $F_1 \notin \mathcal{F}$. $\mathcal{F}$ being maximal, $\mathcal{F} \cup \{F_1\}$ is not a forest, that is F_1 must overlap with at least one biggest subgraph in $\mathcal{F}$, say F_2. From (A.1) and (A.2) $F_1 \cup F_2$ is divergent, since F_1 is maximal, $F_1 \cup F_2 = G$ and G has the structure of Fig. 9 (respectively Fig. 10) if G is a biped (respectively a quadruped). By writing the complete 3- (respectively 2-)particles reducibility of G, the possible biggest subgraphs in $\mathcal{F}$ are obtained in cutting some set of 3 (respectively 2) lines as in Fig. 11 (respectively Fig. 12).

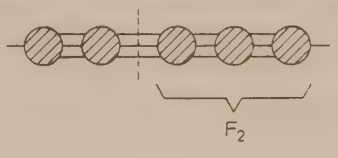

Fig. 11

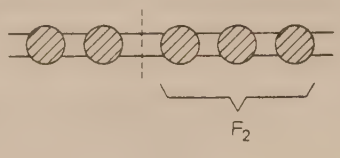

Fig. 12

Let $d(G)$ be the number of maximal closed divergent forests of G, and d_n the maximal value of $d(G)$ over all graphs G with n vertices. Since there is at least one vertex in each maximal divergent closed subgraph and in each bubble of Figs. 11 and 12, we may conclude in both cases i) and ii) that

$$d_n \leqq \sum_{n'=1}^{n-1} d_{n'} d_{n-n'} . \tag{A.3}$$

Lemma A.1.

$$d_n \leqq 4^n . \tag{A.4}$$

Proof. Putting $e_n = \sum_{n'=1}^{n-1} e_{n'} e_{n-n'}$ with $e_1 = 1$, and defining

$$e(z) = \sum_{n=1}^{\infty} e_n z^n ,$$

we have $e^2(z) = e(z) - z$, or $e(z) = 1/2[1 - (1-4z)^{1/2}]$, which is analytic for $|z| < 1/4$. It is easy to verify, by computing the n^{th} derivative of $e(z)$, that $e_n \leqq 4^n$. Now from (A.3), $d_n \leqq e_n$, which proves the lemma.

Finally any closed divergent forest is a subset of a maximal closed divergent forest, which contains at most n subgraphs by Lemma C.1. Using Lemma A.1 we may conclude:

Lemma A.2. *Let $v(G)$ be the number of closed divergent forests of a graph G of Φ_4^4, with n vertices. Then*

$$v(G) \leqq 8^n . \tag{A.5}$$

Appendix B

This appendix is devoted to the proof of Lemma III.6. We fix a given closed divergent forest $\mathcal{F}$, with s elements. We number in an arbitrary way the elements

F_j of $\mathscr{F}\cup\{G\}$ by an index $j\in J=\{1,\ldots,s+1\}$. A spanning tree S of G is called an $\mathscr{F}$-tree of G if $S\cap F_j$ is a spanning tree of F_j for every $j\in J$. Any $\mathscr{F}$-tree of G is defined as well by the family $\{S_j\}$, $j\in J$, where $S_j=S\cap F_j/\mathscr{F}$ is a spanning tree of $F_j/\mathscr{F}$.

Then to every permutation $\sigma\in\mathfrak{S}_G^{\mathscr{F}}$ we associate an $\mathscr{F}$-tree $S(\sigma)$ by defining the corresponding family $\{S_j(\sigma)\}$

$$S_j(\sigma)=\{\sigma(i)|\sigma(i)\in F_j/\mathscr{F}\,;\,L(G_i^\sigma\cap F_j/\mathscr{F})=L(G_{i-1}^\sigma\cap F_j/\mathscr{F})\}\,, \tag{B.1}$$

and we define

$$k_j(\sigma)=\operatorname*{Sup}_{\sigma(i)\in S_j(\sigma)} i\,, \tag{B.2}$$

$$\tilde{\mathfrak{S}}_G^{\mathscr{F}}=\{\sigma|\sigma\in\mathfrak{S}_G^{\mathscr{F}},k_1(\sigma)<k_2(\sigma)<\ldots<k_{s+1}(\sigma)\}\,. \tag{B.3}$$

Since there are $(s+1)!$ ways to choose the arbitrary numbering of the subgraphs F_j, and since

$$(s+1)!/s!=s+1\leqq n(G)+1\leqq 2^{n(G)}\,, \tag{B.4}$$

we see that Lemma III.6 is a direct consequence of the existence of a constant K_4 such that

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_G^{\mathscr{F}}}\left[\prod_{\substack{i\\ \sigma(i)\in S(\sigma)}}\frac{1}{\tilde{N}_i(\sigma)}\right]\cdot\left[\prod_{\substack{i\\ \sigma(i)\notin S(\sigma)}}\frac{1}{\tilde{N}_i(\sigma)}\right]\leqq(K_4/2)^{n(G)}\,. \tag{B.5}$$

In order to prove (B.5), we introduce some new definitions: Given two subgraphs F and F', with $F'\subsetneq F$ denote by $N_F(F')$ the number of internal lines of F which are external lines of F'. Given an $\mathscr{F}$-tree S (defined by the family $\{S_j\}$, $j\in J$) and a permutation $\sigma\in\mathfrak{S}_S$, we define $k_j(\sigma)$ by (B.2) and we put

$$\tilde{\mathfrak{S}}_S=\{\sigma|\sigma\in\mathfrak{S}_S,k_1(\sigma)<k_2(\sigma)<\ldots<k_{s+1}(\sigma)\}\,, \tag{B.6}$$

$$M(S_i^\sigma)=1+\sum_{\substack{j\in J\\ k_j(\sigma)>i}}[N_{S_j}(S_j\cap S_i^\sigma)+1]\,, \tag{B.7}$$

where S_i^σ is defined for $\sigma\in\mathfrak{S}_S$, just like G_i^σ for $\sigma\in\mathfrak{S}_G$.

$$M_S(\sigma)=\prod_{i=1}^{\ell(S)}1/M(S_i^\sigma)\,. \tag{B.8}$$

Finally to any permutation $\sigma\in\tilde{\mathfrak{S}}_G^{\mathscr{F}}$ we associate the permutation $\psi(\sigma)=\sigma'\in\tilde{\mathfrak{S}}_{S(\sigma)}$ naturally induced by σ: $\sigma'(i')=\sigma(i)\Leftrightarrow\sigma(i)$ is the i'th line belonging to $S(\sigma)$. That is

$$\sigma'(i')=\sigma(i)\Leftrightarrow\sigma(i)\in S(\sigma)\,;\,|\{i''<i|\sigma(i'')\in S(\sigma)\}|=i'-1\,. \tag{B.9}$$

Then we have:

Lemma B.1. *The left-hand side of* (B.5) *is bounded by*

$$3^{n(G)}\sum_{\substack{S\\ S\,\mathscr{F}\text{-tree of }G}}\sum_{\sigma'\in\tilde{\mathfrak{S}}_S}M_S(\sigma')\sum_{\substack{\sigma\in\tilde{\mathfrak{S}}_G^{\mathscr{F}}\\ S(\sigma)=S\\ \psi(\sigma)=\sigma'}}\prod_{\substack{i\\ \sigma(i)\notin S(\sigma)}}1/\tilde{N}_i(\sigma)\,. \tag{B.10}$$

Proof. We have $\tilde{N}_i(\sigma) \geqq 1$ since $\sigma \in \mathfrak{S}_G^{\mathscr{F}}$. Moreover if $\sigma'(i') = \sigma(i) \in S$ and $k_j(\sigma') > i'$, we have

$$G_i^{\sigma} \cap F_j / \mathscr{F} \neq F_j / \mathscr{F}$$

and

$$N(G_i^{\sigma} \cap F_j / \mathscr{F}) \geqq \mathrm{Sup}\{1, N_{S_j}(S_j \cap S_{i'}^{\sigma'})\}.$$

Thus

$$2N(G_i^{\sigma} \cap F_j / \mathscr{F}) \geqq N_{S_j}(S_j \cap S_{i'}^{\sigma'}) + 1,$$

and

$$\sigma'(i') = \sigma(i) \in S \Rightarrow 3\tilde{N}_i(\sigma) \geqq M(S_{i'}^{\sigma'}). \tag{B.11}$$

Therefore

$$\prod_{\substack{i \\ \sigma(i) \in S(\sigma)}} 1/\tilde{N}_i(\sigma) \leqq 3^{\ell(S(\sigma))} M_{S(\sigma)}(\psi(\sigma)). \tag{B.12}$$

Since $S(\sigma)$ is a spanning tree of G, $\ell(S(\sigma)) = n(G) - 1$, which achieves the proof of Lemma B.1.

Lemma B.2.

$$\forall S, \sigma', \sum_{\substack{\sigma \in \mathfrak{S}_G^{\mathscr{F}} \\ S(\sigma) = S \\ \psi(\sigma) = \sigma'}} \prod_{\substack{i \\ \sigma(i) \notin S(\sigma)}} \frac{1}{\tilde{N}_i(\sigma)} \leqq 2^{\ell(G)-1}. \tag{B.13}$$

Proof. By induction, let us assume that we have chosen $\sigma(1), \ldots, \sigma(i-1)$. We must then choose a line $\sigma(i)$ such that $S(\sigma) = S$ and $\psi(\sigma) = \sigma'$. We get a factor 2 by deciding whether $\sigma(i)$ belongs to S or not.

i) If $\sigma(i) \in S$, $\sigma(i)$ is determined by σ'.

ii) If $\sigma(i) \notin S$, $\sigma(i)$ belongs to one $F_j / \mathscr{F}$ and adds a loop to $G_{i-1}^{\sigma} \cap F_j / \mathscr{F}$. In this case there is at most $\tilde{N}_{i-1}(\sigma)/2$ possible choices for $\sigma(i)$, and we have $\tilde{N}_i(\sigma) = \tilde{N}_{i-1}(\sigma) - 2$, hence $\tilde{N}_{i-1}(\sigma) \geqq 4$ (since there is no vacuum graph) and $\tilde{N}_i(\sigma) \geqq \tilde{N}_{i-1}(\sigma)/2$. Therefore the number of possible choices is bounded by $\tilde{N}_i(\sigma)$.

At the last step, $\sigma(\ell(G))$ is completely determined, which achieves the proof of Lemma B.2.

Now we want to evaluate $\sum_{\sigma' \in \mathfrak{S}_S} M_S(\sigma')$. Before doing it we construct a one-to-one correspondence ξ_S between the lines of a given connected tree S, and the lines of the chain-like tree D, with $n(D) = n(S)$ (Fig. 13): drawing S on a plane, we "turn around it", starting from an arbitrary extremal line, and we number the lines in the order we meet them the first time. The lines of D are numbered in the same way, and ξ_S associates the lines with the same number. Then we have:

Lemma B.3. *For any (connected or disconnected) subgraph R of S*

$$N_S(R) + 1 \geqq c(\xi_S(R)), \tag{B.14}$$

where $c(\xi_S(R))$ is the number of connected components of $\xi_S(R)$.

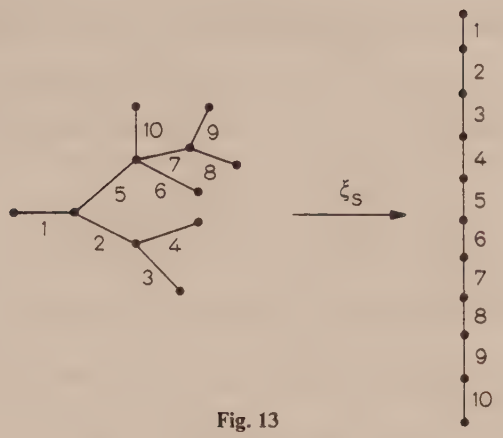

Fig. 13

Proof. Lemma B.3 is trivial for $n(S)=2$ and 3. Inductively, we assume it for $n(S)\leqq n$, and we consider a given S with $n(S)=n+1$ [thus $\ell(S)=n$]. Then the line numbered as n in the preceding numbering is certainly an extremal line of S. Let R be a given subgraph of S.

a) $n\notin R$. Applying the induction hypothesis to R as a subgraph of $S'=S-\{n\}$ proves (B.14).

b) $n\in R$ and $n-1\in R$. Applying the induction hypothesis to $R'=R-\{n\}$ and $S'=S-\{n\}$ gives (B.14) since $N_S(R)=N_{S'}(R')$; $c(\xi_S(R))=c(\xi_{S'}(R'))$.

c) $n\in R$, $n-1\notin R$ and $n-1$ connects with R. Applying the induction hypothesis to $R'=R-\{n\}$ and $S'=S-\{n,n-1\}$ gives (B.14) since

$$N_S(R)\geqq N_{S'}(R')+1 \quad \text{and} \quad c(\xi_S(R))=c(\xi_{S'}(R'))+1\,.$$

d) $n\in R$, $n-1\notin R$ and $n-1$ does not connect with R. Then $n-1$ is extremal in S and $E\cap R=\emptyset$, if E is the set of lines of S which connect with $n-1$. Moreover $E\not\ni 1$.

Either all the lines in E connect with R. Since there exists one line $j\in E$ with

$$j+1\in E\cup\{n-1\}\,,$$

we can contract this line and renumber the tree $S'=S/\{j\}$ by the same procedure. Applying the induction hypothesis to S' and $R'=R\cup\{j\}/\{j\}$ gives (B.14) since $N_S(R)=N_{S'}(R')$ and

$$c(\xi_S(R))=c(\xi_{S'}(R'))\,.$$

Or else there exists a line k in E which does not connect with R. Then $k+1$ connects with k (trivially if k is not extremal, because k connects with $n-1$ if k is extremal), hence $k+1\notin R$. Applying the induction hypothesis to $S'=S/\{k\}$ and $R'=R\cup\{k\}/\{k\}$ gives (B.14) since again $N_S(R)=N_{S'}(R')$ and $c(\xi_S(R))=c(\xi_{S'}(R'))$.

Lemma B.4. *Let D be the chain-like tree with n vertices. Then if we put*

$$\Delta_n=\sum_{\sigma\in\mathfrak{S}_D}\prod_{i=1}^{n-1}1/c(D_i^\sigma)\,, \tag{B.15}$$

we have

$$\Delta_n\leqq 4^n\,. \tag{B.16}$$

Proof. Let $G_1, \ldots, G_q$ be q disjoint graphs (each of them being connected or disconnected), with $\ell_1, \ldots, \ell_q$ internal lines. Let G be their union, with $\ell = \sum_{j=1}^{q} \ell_j$ internal lines. For any $\sigma \in \mathfrak{S}_G$, let $\psi_j(\sigma)$ be the permutation naturally induced on the lines of G_j by (B.9). Then, given $\sigma_1 \in \mathfrak{S}_{G_1}, \ldots, \sigma_q \in \mathfrak{S}_{G_q}$, we have

$$\sum_{\substack{\sigma \in \mathfrak{S}_G \\ \psi_j(\sigma) = \sigma_j \forall j}} \prod_{i=1}^{\ell} 1/c(G_i^{\sigma}) = \prod_{j=1}^{q} \prod_{i=1}^{\ell_j} 1/c[(G_j)_i^{\sigma_j}]. \tag{B.17}$$

This can be proved by induction on ℓ: it is true for $\ell = 1$ and we have

$$\sum_{\substack{\sigma \in \mathfrak{S}_G \\ \psi_j(\sigma) = \sigma_j}} \prod_{i=1}^{\ell} 1/c(G_i^{\sigma}) = \frac{1}{c(G)} \sum_{j=1}^{q} \sum_{\substack{\sigma \in \mathfrak{S}_G \\ \psi_j(\sigma) = \sigma_j \\ \sigma(\ell) = \sigma_j(\ell_j)}} \prod_{i=1}^{\ell-1} 1/c(G_i^{\sigma}).$$

If we assume (B.17) for $G - \{\sigma(\ell)\} = \left[\bigcup_{\substack{k=1 \\ k \neq j}}^{q} G_k\right] \cup [G_j - \sigma_j(\ell_j)]$, we get

$$\sum_{\substack{\sigma \in \mathfrak{S}_G \\ \psi_j(\sigma) = \sigma_j}} \prod_{i=1}^{\ell} 1/c(G_i^{\sigma}) = \frac{1}{c(G)} \sum_{j=1}^{q} c(G_j) \prod_{k=1}^{q} \prod_{i=1}^{\ell_k} 1/c[(G_k)_i^{\sigma_k}],$$

which proves (B.17) for G since $c(G) = \sum_{j=1}^{q} c(G_j)$. Now since $c(D_{n-1}^{\sigma}) = c(D) = 1$, we have

$$\Delta_n = \sum_{i=1}^{n-1} \sum_{\substack{\sigma_1 \in \mathfrak{S}_{D_1(i)} \\ \sigma_2 \in \mathfrak{S}_{D_2(i)}}} \sum_{\substack{\sigma \in \mathfrak{S}_D \\ \psi_1(\sigma) = \sigma_1 \\ \psi_2(\sigma) = \sigma_2 \\ \sigma(n-1) = i}} \prod_{i'=1}^{n-2} 1/c(D_{i'}^{\sigma}), \tag{B.18}$$

where $D_1(i)$ is the chain $\{1, \ldots, i-1\}$ and $D_2(i)$ the chain $\{i+1, \ldots, n-1\}$. By (B.17), with the convention $\Delta_1 = 1$, (B.18) may be written

$$\Delta_n = \sum_{i=1}^{n-1} \Delta_i \Delta_{n-i}. \tag{B.19}$$

Therefore, by the proof of Lemma A.1, $\Delta_n = e_n$ and Lemma B.4 is proved.

Lemma B.5.

$$\sum_{\sigma \in \mathfrak{S}_S} M_S(\sigma) \leqq 4^{n(G)}. \tag{B.20}$$

Proof. Given an $\mathscr{F}$-tree S defined by the family $\{S_j\}$, $j \in J$, we take the correspondence ξ_{sj} between the lines of each S_j and the lines of the related chain D_j with $\ell(D_j) = \ell(S_j)$. Then we renumber the lines of $D_j = \{1, \ldots, \ell(D_j)\}$ as $\{1 + t_j, \ldots, \ell(D_j) + t_j\}$, where $t_j = \sum_{j' < j} \ell(D_{j'})$. By connecting the chains $D_1, \ldots, D_{s+1}$, we build the chain D with $\ell(D) = \ell(S) = n(G) - 1$ and we get a one-to-one correspondence τ_S between the lines of S and the lines of D.

For $\sigma \in \tilde{\Xi}_S$, we have from (B.6) $k_j(\sigma) \leqq i \Leftrightarrow j \leqq j_{\max}(i)$. Therefore

$$\bigcup_{\substack{j \in J \\ k_j(\sigma) \leqq i}} S_j \subseteqq S_i^\sigma$$

and

$$c\left[\tau_S\left(\bigcup_{\substack{j \in J \\ k_j(\sigma) \leqq i}} S_j\right)\right] = c\left(\bigcup_{j=1}^{j_{\max}(i)} D_j\right) = 1. \tag{B.21}$$

For $k_j(\sigma) > i$ we have from Lemma B.3

$$1 + N_{S_j}(S_j \cap S_i^\sigma) \geqq c[\tau(S_j \cap S_i^\sigma)]. \tag{B.22}$$

Using (B.21) and (B.22) in the definition (B.7), we have

$$\forall \sigma \in \tilde{\Xi}_S, \qquad M(S_i^\sigma) \geqq c(\tau_S(S_i^\sigma)). \tag{B.23}$$

Since τ_S is a one-to-one correspondence between the lines of S and the lines of D, (B.23) and Lemma B.4 achieve the proof of (B.20).

Finally Lemmas B.1, B.2, and B.5 achieve the proof of formula (B.5) and Lemma III.6

Appendix C

In this appendix we prove Theorem II. Let us remark first:

Lemma C.1.

$$\forall G, f_2(G) \leqq n(G). \tag{C.1}$$

Proof. For any forest $\mathcal{F}$ of quadrupeds of G, and any $F \in \mathcal{F}$, we have

$$n(F) = n(F/\mathcal{F}) + \sum_{F' \in \mathcal{M}_{\mathcal{F}}(F)} [n(F') - 1],$$

and $\forall F' \in \mathcal{F}$, $n(F'/\mathcal{F}) \geqq 2$. A simple induction gives

$$f(\mathcal{F}) = q(\mathcal{F}) \leqq n(G) - 1. \tag{C.2}$$

From the definition (II.16) of $f_2(G)$, (C.2) proves (C.1).

Now let $\gamma(N, n, f)$ be the number of graphs G with $N(G) = N$, $n(G) = n$ and $f_2(G) = f$. To any such graph we associate a given arbitrary forest $\mathcal{F}(G)$ of quadrupeds of G, such that $G \notin \mathcal{F}(G)$ and $f(\mathcal{F}(G))$ is maximal with this restriction, that is

$$f(\mathcal{F}(G)) = f' = f - 1, \quad \text{if} \quad N(G) \leqq 4,$$
$$f(\mathcal{F}(G)) = f' = f, \quad \text{if} \quad N(G) > 4,$$

and in any case, by (C.2)

$$f \leqq f' + 1 \leqq n. \tag{C.3}$$

We call $\lambda(G)$ the number of elements in a maximal nest of $\mathcal{F}(G)$, i.e. in a maximal subset of $\mathcal{F}(G)$ totally ordered by inclusion. Then we separate $\mathcal{F}(G)$ into layers $\mathcal{L}_i$, $1 \leqq i \leqq \lambda(G)$, which realize a partition of $\mathcal{F}(G)$

$$\begin{aligned} \mathcal{L}_{\lambda(G)} &= \{F | F \in \mathcal{F}(G); B_{\mathcal{F}(G)}(F) = G\} \\ \mathcal{L}_i &= \{F | F \in \mathcal{F}(G); B_{\mathcal{F}(G)}(F) \in \mathcal{L}_{i+1}\} \quad \text{for} \quad 1 \leqq i \leqq \lambda(G) - 1. \end{aligned} \tag{C.4}$$

We also define

$$h_i(G)=|\mathscr{L}_i|, \tag{C.5}$$

$$k_i(G)=\sum_{i'=1}^{i} h_{i'}(G). \tag{C.6}$$

Therefore

$$h_i(G)>0 \quad \text{for} \quad 1\leqq i\leqq\lambda(G), \tag{C.7}$$

$$0<k_1(G)<k_2(G)<\ldots<k_{\lambda(G)}=f'. \tag{C.8}$$

Next we order arbitrarily the disjoint subgraphs in each layer, and we label them in the following way

$$\mathscr{L}_i=\{F_j(G)|j=k_{i-1}+1,\ldots,k_i\}, \tag{C.9}$$

with the conventions

$$k_0=0, \quad h_0=0, \quad \text{and} \quad G=F_{f'+1} \tag{C.10}$$

(There are of course $\prod_{i=1}^{\lambda(G)}[h_i(G)!]$ possible different labelings.) We define

$$n_j(G)=n[F_j(G)/\mathscr{F}(G)], \tag{C.11}$$

$$s_j(G)=|\mathscr{A}_{\mathscr{F}(G)}(F_j(G))|, \tag{C.12}$$

for $1\leqq j\leqq f'+1$. With all these definitions we get

$$\gamma(N,n,f)=\sum_{\lambda}\sum_{\{k_i|i=1,\ldots,\lambda\}}\left[\prod_{i=1}^{\lambda}1/h_i!\right] \cdot\sum_{\{n_j,s_j|j=1,\ldots,f'+1\}}\gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\}), \tag{C.13}$$

where $\gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\})$ is the number of graphs G with $N(G)=N$, $n(G)=n,\ldots,s_j(G)=s_j$. If this number does not vanish, the following relations must be satisfied:

$$\forall j, 1\leqq j\leqq f'+1, \quad 2\leqq n_j\leqq n, \tag{C.14}$$

$$\sum_{j=1}^{f'+1} n_j=n+f', \tag{C.15}$$

$$\sum_{j=k_{i-1}+1}^{k_i} s_j=h_{i-1}; \qquad \sum_{j=1}^{f'+1} s_j=f'. \tag{C.16}$$

Now the number of sequences $\{k_i\}$ satisfying (C.8) is equal to $\binom{f'-1}{\lambda-1}$, and

$$\sum_{\lambda=1}^{f'}\binom{f'-1}{\lambda-1}=2^{f'-1}\leqq 2^n. \tag{C.17}$$

By similar standard combinatoric arguments, the number of sets $\{n_j\}$ (respectively $\{s_j\}$) satisfying (C.14), (C.15) [respectively (C.16)] is bounded by 2^n (respectively $2^{2f'}$). Therefore the proof of Theorem II is achieved by the following lemma:

Lemma C.2. *$\forall \lambda$, $\{k_i\}$, $\{n_j\}$, $\{s_j\}$ satisfying the preceding relations, we have*

$$\left[\prod_{i=1}^{\lambda} \frac{1}{h_i!}\right] \gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\}) \leqq K_8^n \frac{(n!)^2}{f!} n^{N/2} C(N), \tag{C.18}$$

K_8 being a positive constant.

Proof. The number of graphs with N external legs and n internal vertices is bounded by $(4n+N-1)!!$ [1]; therefore it is bounded by $K_9^n (n!)^2 n^{N/2} C(N)$ for any N, and by $K_9^n (n!)^2$ for $N=4$, K_9 being a positive constant.

Hence it is possible to bound the left hand side of (C.18) by

$$\prod_{i=1}^{\lambda} \left\{ \frac{1}{h_i!} \prod_{j=k_{i-1}+1}^{k_i} \left[\binom{n - \sum_{j'=1}^{j-1} r_{j'}}{r_j} \binom{h_{i-1} - \sum_{j'=k_{i-1}+1}^{j-1} s_{j'}}{s_j} K_9^{n_j} (n_j!)^2 (4!)^{s_j} \right] \right\}$$
$$\cdot K_9^{n_{f'+1}} (n_{f'+1}!)^2 n_{f'+1}^{N/2} C(N) (4!)^{s_{f'+1}}, \tag{C.19}$$

where $r_j = n_j - s_j$. Indeed we estimate the left-hand side of (C.18) by building successively

$$F_1/\mathcal{F}, \ldots, F_j/\mathcal{F}, \ldots, F_{f'+1}/\mathcal{F}.$$

The binomial coefficients

$$\binom{n - \sum_{j'=1}^{j-1} r_{j'}}{r_j} \quad \text{and} \quad \binom{h_{i-1} - \sum_{j'=k_{i-1}+1}^{j-1} s_{j'}}{s_j}$$

choose respectively the ordinary vertices and the reduction vertices which belong to $F_j/\mathcal{F}$; the terms $K_9^{n_j}(n_j!)^2$ or $K_9^{n_{f'+1}}(n_{f'+1}!)^2 n_{f'+1}^{N/2} C(N)$ bound the number of graphs made with these vertices. The factors $(4!)^{s_j}$ take into account the possible identifications between the 4 external legs of F_j, and the 4 lines attached to the reduction vertex corresponding to F_j in $B_{\mathcal{F}(G)}(F_j)/\mathcal{F}$.

But, using (C.15) and (C.16), (C.19) is reduced to

$$K_9^{n+f'} [4!]^{f'} n_{f'+1}^{N/2} C(N) n! \prod_{j=1}^{f'+1} n_j! \binom{n_j}{s_j}. \tag{C.20}$$

Therefore by (C.14), (C.15), and (C.3), since $\binom{n_j}{s_j} \leqq 2^{n_j}$, (C.20) is bounded by

$$[4(4!)K_9^2]^n n^{N/2} C(N) n! \prod_{j=1}^{f'+1} n_j! \tag{C.21}$$

By (C.14) and (C.15) again, $\prod_{j=1}^{f'+1} n_j!$ is a product of $n+f'$ integers smaller than n, among which at least $f'+1$ are 1 and $f'+1$ are 2. Therefore by Stirling's formula, and (C.3)

$$\prod_{j=1}^{f'+1} n_j! \leqq 2^{f'+1} n^{n-f'-2} \leqq (2e)^n \frac{n!}{n^{f'+1}} \leqq (2e)^n \frac{n!}{f!}, \tag{C.22}$$

which achieves the proof of (C.18) and Theorem II.

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The Perturbation Series for Φ_3^4 Field Theory is Divergent

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Abstract. We prove in a rigorous way the statement of the title.

I. Introduction

At least since a paper by Dyson [1], the perturbation expansion for field theories like Φ^4 or Q.E.D. is commonly believed to be a divergent series. However it has been periodically noticed [2] [3] [4] that this statement is rigorously proved only for the simplest models, i.e. $[P(\Phi)]_2$ [5].

In the case of euclidean Φ_3^4, we know the results of constructive field theory [6] and the Borel summability of the perturbation expansion [7]. Yet there is no proof that this series is not actually convergent. The difficulty which prevents Jaffe's method [5] from working for Φ_3^4 is the change in the signs of some amplitudes, due to the renormalization, which could produce cancellations at each order. We solve this problem by rewriting the usual perturbation expansion in terms of convergent positive amplitudes involving a "dressed" propagator. This method is an iteration of the procedure used by Hepp [3] for the regularized version of the model.

More generally, the control of signs in renormalized perturbation series could allow one to go beyond the recent results on the Borel transformed series for Φ^4 theories [8] [9]. Extensions of our method might then provide rigorous results on the presence—or absence—of singularities on the real axis of the Borel plane ("instantons" or "renormalons" [10]).

II. Proof

We consider massive scalar bosons (the mass-scale is fixed by taking $m = 1$) self-interacting via the lagrangian density $\mathscr{L}_I = -\frac{\lambda}{4!} : \Phi^4 :$ in an euclidean space-time with three dimensions. The formal series in λ defining a given N-points connected Schwinger function S_N is given in terms of Feynman graphs G (with $N(G)$ external lines, $n(G)$ vertices) by:

$$S_N(p, \lambda) = \sum_{\substack{G \\ N(G) = N}} A_G^R(p, \lambda), \tag{1}$$

0010-3616/82/0083/0077/$01.20

where $A_G^R(p,\lambda)=(-\lambda)^{n(G)}I_G^R(p)$ is the renormalized amplitude corresponding to G, and $p=\{p_1,\ldots,p_N\}$ is the set of external euclidean momenta.

As in [7] we use Zimmermann's scheme to renormalize by subtracting at vanishing external momenta [11]. (Actually our proof does not depend on the subtraction point.) Now the only renormalization to be performed corresponds to the graph (or subgraph) in Fig. 1, which we call the "blob" in the rest of the paper. The bare propagator is:

$$P_0=\frac{1}{p^2+1}, \tag{2}$$

and the blob renormalized amplitude is given by:

$$B_0(p,\lambda)=\frac{\lambda^2}{3}\int d^3k\,d^3k'\left[\frac{1}{(k^2+1)((k+k')^2+1)((p+k')^2+1)}\right.$$
$$\left.-\frac{1}{(k^2+1)((k+k')^2+1)(k'^2+1)}\right]. \tag{3}$$

We introduce an operation $\mathscr{S}$ on the graphs $G:\mathscr{S}(G)$ is the graph obtained by replacing in G every maximal chain of blobs by a single line (see an example in Fig. 2). In order to add the amplitudes of the graphs G which have the same $\mathscr{S}(G)$, we define inductively dressed propagators P_i and amplitudes B_i by:

$$P_i(p,\lambda)=P_{i-1}+P_{i-1}B_{i-1}P_{i-1}+\ldots=\frac{1}{p^2+1-\sum_{j=0}^{i-1}B_j(p,\lambda)}, \tag{4}$$

$$B_i(p,\lambda)=\frac{\lambda^2}{3}\int d^3k\,d^3k'\left[P_i(k)P_i(k+k')P_i(p+k')\right.$$
$$\left.-P_{i-1}(k)P_{i-1}(k+k')P_{i-1}(p+k')\right]. \tag{5}$$

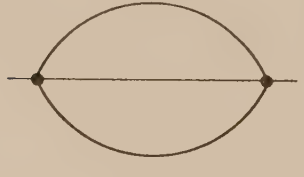

Fig. 1

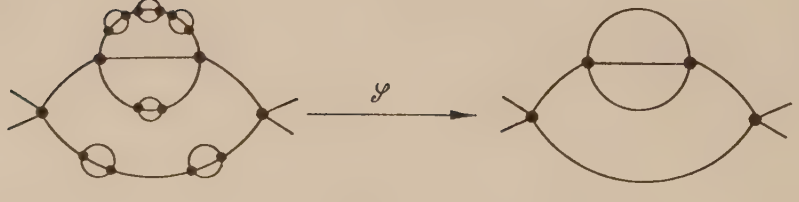

Fig. 2

$B_i(p,\lambda)$ is the sum of the amplitudes built from a blob with P_{i-1} propagators by inserting an arbitrary number of B_{i-1} on its three internal lines, the total number of insertions being not zero.

Next we associate to any graph G an amplitude $A_G^i(p,\lambda)$ in the following way. Let $\tilde{G}$ be the graph obtained by reducing every blob in G to a two-line vertex. Then $A_G^i(p,\lambda)$ is the amplitude corresponding to $\tilde{G}$ with a propagator P_i for each line, a factor B_i for each reduction vertex and a factor $(-\lambda)$ for each ordinary four-line vertex (in particular $A_G^0(p,\lambda) = A_G^R(p,\lambda)$). With these definitions we obtain:

$$\sum_{\substack{G' \\ \mathscr{S}(G')=G}} A_{G'}^i(p,\lambda) = A_G^{i+1}(p,\lambda). \tag{6}$$

Equation (6) is trivial from definition (4) if G contains no blob ($\mathscr{S}(G) = G$), and takes into account definition (5) if G contains blobs ($\mathscr{S}(G) \neq G$).

Lemma 1. *There exist three positive constants M_1, ρ_1 and $\varepsilon(\varepsilon < 1)$ such that, for $|\lambda| < \rho_1$, $B_i(p,\lambda)$ is analytic in λ, and:*

$$\forall i \geqq 0, \quad |B_i(p,\lambda)| < (p^2+1)^{\varepsilon} \cdot (\tfrac{1}{3})^{i+1} \cdot 2M_1 |\lambda^2|. \tag{7}$$

Proof. Inequality (7) is well-known for B_0. Let us assume it is true for B_j, $\forall j \leqq i-1$. Then:

$$\forall k \leqq i-1, \quad \left| \sum_{j=0}^{k} B_j(p,\lambda) \right| \leqq (p^2+1)^{\varepsilon} M_1 |\lambda^2| \leqq \frac{p^2+1}{2}, \tag{8}$$

if we take

$$\rho_1^2 \leqq \frac{1}{2M_1}. \tag{9}$$

Therefore $\forall k \leqq i$:

$$\frac{2/3}{p^2+1} \leqq \frac{1}{(p^2+1)(1+M_1|\lambda^2|)} \leqq |P_k(p,\lambda)| \leqq \frac{1}{(p^2+1)(1-M_1|\lambda^2|)} \leqq \frac{2}{p^2+1} \tag{10}$$

and

$$|P_i - P_{i-1}| = |B_{i-1} P_i P_{i-1}| \leqq \frac{8M_1|\lambda^2|(\frac{1}{3})^i}{(p^2+1)^{2-\varepsilon}} \leqq 2|P_{i-1}|. \tag{11}$$

We write equation (5) as

$$B_i = \frac{\lambda^2}{3} \int [(P_i - P_{i-1})(P_i - P_{i-1})(P_i - P_{i-1}) - 3P_i P_{i-1}(P_i - P_{i-1})], \tag{12}$$

and we find

$$|B_i| \leqq 8M_1|\lambda^2| \frac{28|\lambda^2|}{3} \left(\frac{1}{3}\right)^i \int \frac{d^3k\, d^3k'}{(k^2+1)^{2-\varepsilon}((k+k')^2+1)((p+k')^2+1)}. \tag{13}$$

The integral in (13) is known to be bounded by a constant M_2. Choosing

$\rho_1^2 < \mathrm{Inf}\left(\frac{1}{2M_1}, \frac{1}{4.28 M_2}\right)$ proves the lemma.

From Lemma 1, the infinite sum in (6) is absolutely convergent for $|\lambda| < \rho_1$. Furthermore:

$$P_i(p,\lambda) \xrightarrow[i\to\infty]{} P_\infty(p,\lambda). \tag{14}$$

For $|\lambda| < \rho_1$, $P_\infty(p,\lambda)$ is analytic in λ, real positive for λ real, and satisfies:

$$\frac{1}{1+M_1|\lambda^2|} P_0(p) \leqq |P_\infty(p,\lambda)| \leqq \frac{1}{1-M_1|\lambda^2|} P_0(p). \tag{15}$$

Let G be a graph without blob ($\mathscr{S}(G) = G$). We call $\Gamma_r(G)$ the set of graphs G' such that $\mathscr{S}^r(G') = G, r$ being the smallest such integer. Then we have from (6):

$$\sum_{G' \in \Gamma_r(G)} A^0_{G'} = A^r_G - A^{r-1}_G, \quad \forall r \geqq 1 \tag{16}$$

and from (14) and (16), with $\Gamma(G) = \bigcup_{r=0}^{\infty} \Gamma_r(G)$:

$$\sum_{G' \in \Gamma(G)} A^0_{G'}(p,\lambda) = A^\infty_G(p,\lambda), \tag{17}$$

where A^∞_G is defined by replacing each propagator in G by P_∞. Again the sum (17) is absolutely convergent.

Remark. The same work can be done with the modulus of the amplitudes. Definitions (4) and (5) can be replaced by:

$$Q_i = \frac{1}{p^2 + 1 + \sum_{j=0}^{i-1} C_j(p,\lambda)}, \tag{18}$$

$$C_0 = |B_0|, \quad C_i = \frac{|\lambda^2|}{3} \int [Q_i Q_i Q_i - Q_{i-1} Q_{i-1} Q_{i-1}], \quad \forall i \geqq 1. \tag{19}$$

Inequality (7) is also valid for the C_i's and we find similarly:

$$\sum_{G' \in \Gamma(G)} |A^0_{G'}(p,\lambda)| = F^\infty_G(p,\lambda), \tag{20}$$

where F^∞_G is defined by using a factor $|\lambda|$ for each vertex, and a propagator Q_∞ for each line of G, which satisfies for $|\lambda| < \rho_1$:

$$P_0(p) \leqq Q_\infty(p,\lambda) \leqq \frac{1}{1-M_1|\lambda^2|} P_0(p). \tag{21}$$

Lemma 2. *There exists a positive K, depending only on the external momenta $p_1, \ldots, p_N$, such that for any graph G without blob and any negative value of λ, with $-\rho_1 < \lambda < 0$, we have:*

$$A^0_G(p,\lambda) > (-\lambda K)^{n(G)}. \tag{22}$$

Proof. The proof follows the same lines as in [5]. In the Schwinger parametric representation (with the notations of [9]):

$$A_G^0 = (-\lambda)^n \int_0^\infty \pi d\alpha U^{-3/2} e^{-\Sigma\alpha - V/U} > (-\lambda)^n \int_0^1 \pi d\alpha U^{-3/2} e^{-\Sigma\alpha - V/U}. \tag{23}$$

U (respectively V) is a sum of monomials in the α's corresponding to the one-trees (respectively two-trees) of G. The number of one-trees is bounded by $2^{\ell(G)}$ where the number $\ell(G)$ of internal lines of G is:

$$\ell(G) = 2n(G) - \frac{N(G)}{2}. \tag{24}$$

Any two-tree is obtained by removing a line from some one-tree. Therefore it is easy to see that for $0 \leqq \alpha_i \leqq 1, \forall i$, we have $U^{-3/2} > 2^{-3/2\,\ell}$ and $\frac{V}{U} > (n-1)M^2$, where M^2 is a bound on the invariants built from the external momenta. Lemma 2 follows, with $K = \frac{1}{8}\exp(-2-M^2)$.

Theorem. *The expansion* (1), *as a power series in* λ, *has a vanishing radius of convergence.*

Proof. If the expansion (1) has a non-vanishing radius of convergence ρ_2, then there exists a positive constant M_3 such that, for $|\lambda| \leqq \frac{1}{2}\rho_2$, the partial sums of the expansion are uniformly bounded by M_3, that is:

$$\forall q \geqq 1, \quad \forall |\lambda| \leqq \tfrac{1}{2}\rho_2, \quad |S_{N_q}| = \left| \sum_{\substack{G \\ N(G)=N \\ n(G)\leqq q}} A_G^0 \right| \leqq M_3. \tag{25}$$

However for λ real negative, the A_G^∞'s are real positive. From (17) we have for $-\operatorname{Inf}(\rho_1, \frac{1}{2}\rho_2) \leqq \lambda \leqq 0$:

$$|S_{N_q}| = \left| \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^\infty - \sum_{\substack{G' \\ G'\in\Gamma(G) \\ n(G)\leqq q \\ n(G')>q}} A_{G'}^0 \right| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^\infty - \sum_{\substack{G' \\ G'\in\Gamma(G) \\ n(G)\leqq q \\ G'\neq G}} |A_{G'}^0|. \tag{26}$$

By (15) we have:

$$A_G^\infty \geqq \left(\frac{1}{1+M_1\lambda^2}\right)^{\ell(G)} A_G^0, \tag{27}$$

and by (20) and (21):

$$\sum_{\substack{G'\in\Gamma(G) \\ G'\neq G}} |A_{G'}^0| \leqq \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{\ell(G)} - 1\right] A_G^0. \tag{28}$$

Since by (24) $\ell(G) \leqq 2n(G)$, we find:

$$|S_{N_q}| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^0 \left\{\left(\frac{1}{1+M_1\lambda^2}\right)^{2n(G)} - \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{2n(G)} - 1\right]\right\} \tag{29}$$

It is easy to verify that there exists a constant $M_4, 0 < M_4 < \mathrm{Inf}(\rho_1, \frac{1}{2}\rho_2)$ such that, for $\lambda = -M_4 q^{-1/2}$, we have

$$\left(\frac{1}{1+M_1\lambda^2}\right)^{2n(G)} - \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{2n(G)} - 1\right] \geqq \frac{1}{2}, \quad \forall n(G) \leqq q. \tag{30}$$

Now by an easy adaptation of the argument in [5] there are at least $\left(n - \frac{N}{2}\right)!$ graphs with n vertices and $\mathscr{S}(G) = G$. Therefore we get, using Lemma 2:

$$|S_{N_q}| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G) \leqq q}} \frac{A_G^0}{2} \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)=q}} \frac{A_G^0}{2} \geqq \frac{\left(q - \frac{N}{2}\right)!(K M_4)^q}{2q^{9/2}} \tag{31}$$

which obviously contradicts (25) for q large enough.

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I.4, Commentaire

L'article précédent est dans notre esprit le premier volet d'un programme esquissé dans l'Introduction générale. Un de nos objectifs à long terme est de tenter de prouver la conjecture de 'Hooft-Parisi concernant l'existence de singularités ou "renormalons" sur l'axe réél positif de la transformée de Borel du développement perturbatif dans le cas de ϕ_4^4. Un problème du même genre mais sans doute nettement plus facile est le suivant: dans le cas de ϕ_3^4, prouver que la première singularité de la transformée de Borel en question se trouve sur l'axe réél négatif. En effet d'après les calculs semi-classiques [33], on s'attend à ce que le terme d'ordre n, a_n, de la série de Taylor à l'origine de cette transformée de Borel se comporte comme $(-K)^n$ pour une constante K positive, c'est à dire à ce que:

$$(-1)^n a_n > 0 \text{ si n grand, et } \lim_{n\to\infty} \frac{1}{n} \log\left[(-1)^n a_n\right] = \log K \qquad \text{(I.4.1)}$$

Un tel comportement correspond à une première singularité au point $-\frac{1}{K}$; "premier" ici signifiant plus proche de l'origine. Il nous suffirait d'établir une borne inférieure du type:

$$(-1)^n a_n \geq [K_1]^n \text{ pour n grand} \qquad \text{(I.4.2)}$$

pour $K_1 > 0$, et le résultat cherché serait démontré. Malheureusement la borne inférieure obtenue dans l'article

précédent n'est pas aussi précise que (I.4.2); elle correspond seulement à:

$$\sum_{k=1}^{n} \left| (-K_2)^k \, a_k \cdot \frac{k!}{[n^{k/2}]} \right| \geq K_3^n [n^{n/2}] \qquad \text{(I.4.3)}$$

pour certaines constantes K_2 et K_3 positives. Cette borne est "compatible" avec (I.4.1) ou (I.4.2) mais évidement plus faible puisque (I.4.1) =>(I.4.2) =>(I.4.3). Il est possible qu'à l'aide d'arguments de type "taubérien" on puisse montrer (I.4.2) à partir de (I.4.3), ce qui correspond plus ou moins à prouver que la contribution dominante dans la somme du membre de gauche de (I.4.3) provient presque toujours du dernier terme. [63]. Pour l'instant nous n'avons pu déduire de (I.4.3) que la conséquence évidente de la divergence de la série des perturbations de ϕ_3^4. Il est intéréssant de noter que notre résultat devrait en entraîner d'autres, analogues, pour les séries perturbatives de modèles voisins: le modèle d'Edwards de polymères en dimension 3 [63], et le modèle d'Ising sur réseau, en dimension 3 également [64].

Il pourrait s'avérer intéréssant de combiner notre méthode, qui contrôle les signes provenant de la renormalisation non-triviale de ϕ_3^4 par l'introduction de propagateurs habillés, et la démonstration de T. Spencer [42], qui établit que la méthode de Lipatov s'applique rigoureusement à des versions sur réseau de la théorie ϕ_3^4, au moins en ce qui concerne le calcul de comportements avec la précision (I.4.1).

On pourrait alors éventuellement obtenir non seulement la conjecture énoncée sur les singularités dans le plan de Borel, mais aussi vérifier rigoureusement la valeur de la constante K trouvée dans la référence [33], et donc trouver le rayon exact de convergence de la transformée de Borel dans ce cas.

DEUXIEME PARTIE

Représentation de Mellin complet et Comportements Asymptotiques des Amplitudes de Feynman renormalisées

"L'obscurité qu'on lui reproche ne tient pas à sa nature, qui est d'éclairer, mais à la nuit qu'elle explore."

Saint-John-Perse

II.1, Introduction et Rappels

Comme nous l'avons déja remarqué brièvement (I.3, section IV) le problème principal traité dans la partie précédente, l'évaluation des grands ordres de la théorie perturbative, est assez étroitement relié à un autre problème important de cette théorie, le calcul du comportment asymptotique des amplitudes de Feynman à grand moments. La série d'articles [8-10] forme un tout consacré à cette question par C. de Calan et collaborateurs. C'est pourquoi nous présentons ici quelques explications relatives à l'article [10] qui suit, sous la forme d'un résumé des Références [8-9] principalement.

Nous utiliserons les notations de la partie I.3 sans les rappeler. Un résultat rigoureux très général concernant le comportement asymptotique des amplitudes de Feynman lorsque certains paramètres tendent vers $+\infty$ est le suivant:

Théorème ([8])

Soit $I_G^R(\{a_k\})$ une amplitude de Feynman renormalisée, $\{a_k\}$ représentant l'ensemble des invariants formés à partir des moments externes et des carrés des masses internes. Supposons que l'on dilate par un paramètre réél λ certaines de ces quantités, $\{a_m\}$, les autres $\{a_n\}$ restant fixées, et que l'on fasse tendre λ vers $+\infty$. Alors $I_G^R(\{a_m,a_n\})$ admet un développement asymptotique de la forme:

$$I_G^R(a_m,a_n) = \sum_{p=p_{max}}^{-\infty} \lambda^p \sum_{q=0}^{q_{max}(p)} (\log \lambda)^q g_{pq}(a_m,a_n) \qquad (II.1.1)$$

où p décrit une progression arithmétique de nombres rationels décroissants, et q, pour p fixé, parcourt un nombre fini d'entiers consécutifs de 0 à $q_{max}(p)$.

Le premier résultat du type (II.1.1), qui établit l'existence de développements asymptotiques selon l'échelle des puissances et puissances de logarithmes pour les amplitudes de Feynman est dû à Weinberg [49], mais se limite au cas de la dilatation de tous les moments externes dans une amplitude convergente. De nombreux articles ont suivi, traitant de la détermination explicite du développement (II.1.1) dans divers cas d'intérêt physique, l'accent n'étant pas mis en général sur la rigueur mathématique. La question restait donc posée de trouver une méthode rigoureuse permettant de prouver des résultats très généraux comme le théorème énoncé et de fournir des algorithmes systématiques de calcul de tous les termes

du développement et non de simples règles un peu empiriques pour trouver les premiers termes.

La technique mathématique la mieux adaptée à l'étude de comportements asymptotiques du type (II.1.1) est sans doute celle de la transformée de Mellin, dans laquelle de tels comportements se traduisent par des propriétés de méromorphie dans la variable de Mellin; les valeurs des puissances p correspondent à la position des pôles en transformée de Mellin, les puissances maximales de logarithmes $q_{max(p)}$ correspondent au degré des pôles à une unité prés, et les facteurs g_{pq} correspondent à des prescriptions de résidus. Cette technique mathématique a été utilisée de plus en plus complètement [6-10]. Les auteurs de [7] ont pu retrouver rigoureusement le résultat de Weinberg dans le cas des amplitudes renormalisées selon le procédé BPHZ. Dans la référence [8] un progrès supplémentaire est accompli. Il est clairement dégagé que les règles naïves du comptage des puissances dans les amplitudes ne sont valides que lorsque on a affaire à des intégrand FINE (factorisés dans chaque secteur de Hepp). On comprend plus clairement alors la différence entre les comportments asymptotiques sous une dilatation de tous les moments externes par exemple, et ceux dans lesquels les parties de l'intégrand dans la représentation paramétrique qui sont dilatées ne sont pas FINE. Pour ce dernier type de comportement la technique de la transformée de Mellin simple ne suffit pas, mais l'on peut résoudre le problème en introduisant plusieurs paramètres de Mellin, un pour chaque morceau FINE des parties

dilatées de l'intégrand, puis en découpant l'intégrale en contributions correspondant aux différents secteurs de Hepp. Cette technique permet donc d'étudier rigoureusement des problèmes pour lesquels le comptage des puissances naïf est insuffisant, tels le comportement à la Regge des amplitudes à quatre corps ou les problèmes de limites infrarouges [50-51].

On remarque cependant que la méthode décrite nécessite un double découpage des amplitudes, en morceaux FINE et en secteurs. Cet inconvénient peut être évité en introduisant dès le départ autant de paramètres de Mellin que de monômes dans la représentation paramètrique [9]. On s'aperçoit que l'intégration sur les paramètres α peut être effectuée explicitment et l'on aboutit, dans le cas convergent, à une nouvelle représentation intégrale des amplitudes de Feynman, dite de Mellin complet. Dans cette représentation l'existence de développements du type (II.1.1) devient une conséquence transparente des propriétés de méromorphie de l'intégrand et leur calcul correspond à l'étude purement géométrique de polyèdres convexes dans les variables de Mellin. L'inconvénient principal est le nombre élevé de ces variables: les paramètres α étant au nombre de ℓ et correspondant aux lignes du graphe de Feynman associé, les variables de Mellin sont au nombre d'environ 2^{ℓ}, à comparer favorablement toutefois aux $\ell!$ secteurs de Hepp.

Notre article [10] consiste à établir l'intéréssante propriété, conjecturée en [9], selon laquelle la renormalisation

BPHZ peut s'effectuer, dans le formalisme de Mellin complet, par de simples translations du contour d'intégration laissant l'intégrand inchangé; et nous obtenons la représentation de Mellin complet des amplitudes de Feynman renormalisées, après calcul explicite de ces translations. Ce résultat explique très clairement pourquoi la renormalisation, qui peut changer profondément le détail des développements asymptotiques (II.1.1) d'une amplitude, ne peut pas changer sa structure en puissances et puissances de logarithmes. Comme notre article [10], au contraire de [9], ne contient pas d'indications à ce sujet nous allons rappeler brièvement comment faire l'étude d'un régime asymptotique un peu plus général que celui du théorème énoncé plus haut.

Suivant les notations de [9] nous considérons une dilatation de certains invariants s_k et de certaines masses par des puissances arbitraires d'un paramètre $\lambda \to +\infty$:

$$\begin{aligned} s_k &\to \lambda^{a_k} s_k \\ m_i^2 &\to \lambda^{a_i} m_i^2 \end{aligned} \qquad \text{(II.1.2)}$$

L'amplitude de Feynman renormalisée considérée s'écrit [10]:

$$I_G^R(\lambda, s_k, m_i^2) = \sum_c \mu_c \int_{\Delta_c} M_G(s_k, m_i^2)\ \lambda^{\psi_0} \qquad \text{(II.1.3)}$$

avec $\psi_0 = \sum_k a_k\, y_k + \sum_i a_i\, \phi_i(x_j, y_k)$

(ψ_0 forme linéaire dans les variables de Mellin)
et M_G intégrand de Mellin complet.
La sommation sur les cellules C et les multiplicités μ_c étant finie il suffit de déterminer le comportement asymptotique d'une intégrale $I_C = \int_{\Delta_C} M_G \lambda^{\psi_0}$ puis d'effectuer une sommation finie pour déterminer un développement du type (II.1.1).
Notons z_i les parties réélles des variables de Mellin x_j ou y_k, et considérons, dans l'espace des z, le demi espace mobile $D_p = \{z/\psi_0(z) - p \geq 0\}$ fonction du paramètre p. Il existe une plus grand valeur de p, p_{max}, telle que $\Delta_c \subset D_p$. La cellule Δ_c est un polyèdre convexe dont les bords sont définis par les hyperplans variétés polaires de l' intégrand M_G. Comme $p_{max} = \inf_{\Delta_c} \psi_0$, on obtient aisément une première borne:

$$|I_G| \leq \text{const.} \quad \lambda^{p_{max} + \eta} \quad \forall \eta > 0 \qquad \text{(II.1.4)}$$

Maintenant on peut déplacer le contour d'intégration de I_C en traversant les variétés polaires bordant Δ_c dont l'intersection n'est autre que $\partial\Delta_c \cap \partial D_{p_{max}}$, et celles là seulement. La technique est exactement la même que celle décrite en détail pour la renormalisation dans l'article qui suit. On aboutit donc à de nouvelles intégrales sur des cellules qui ne sont plus incluses dans $D_{p_{max}}$, et des résidus correspondant à des intégrales sur la "face" $\partial\Delta_c \cap \partial D_{p_{max}}$; les nouvelles intégrales obtenues ont un comportement certainement borné par const. $\lambda^{p_{max} - \varepsilon}$ pour $\varepsilon > 0$; d'autre part sur la face $\partial\Delta_c \cap \partial D_{p_{max}}$ $\psi_0 = p_{max}$ est constant, et cette partie de I_c a donc un comportement en $\lambda^{p_{max}}$ $(P(\log \lambda))$

où P est un polynôme, car la prise de résidus, s'exprimant par des dérivations en z, fait descendre des facteurs log λ à cause du terme $\lambda^{\psi_0} = e^{\psi_0 \log\lambda}$. On a donc bien déterminé le début d'un développement asymptotique du type (II.1.1) et il ne reste plus qu'à poursuivre avec les nouvelles intégrales restantes en déplaçant le demi-espace D_p vers des valeurs $p < p_{max}$.

On remarque que dans le cas des dilatations générales (II.1.2), les puissances de λ ne forment plus nécessairement une progression arithmétique décroissante unique, mais une superposition d'un nombre fini de progressions arithmétiques décroissantes dont les raisons ne sont pas nécessairement rationnelles entre elles; il s'agit en tout cas d'un ensemble discret de réels. Il faut donc modifier en ce sens l'énoncé du théorème.

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Renormalization in the Complete Mellin Representation of Feynman Amplitudes

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Abstract. The Feynman amplitudes are renormalized in the formalism of the CM representation. This Mellin-Barnes type integral representation, previously introduced for the study of asymptotic behaviours, is shown to have the following interesting property: in contrast with the usual subtraction procedures, the renormalization leaves the CM integrand unchanged, and only results into translations of the integration path. The explicit CM representation of the renormalized amplitudes is given. In addition, the dimensional regularization and the extension to spinor amplitudes are sketched.

I. Introduction

The complete Mellin (CM) representation of Feynman amplitudes was introduced in [1]. It generalized the multiple Mellin technique [2] in a systematic way which presents great advantages: in [2] indeed, one had first to divide the integration domain of the Schwinger representation into the so-called Hepp sectors [3], so that the desingularization of the integrals was performed in a mixed way, partly with the Hepp variables and partly with the Mellin representation. On the contrary, the CM representation avoids the Hepp sectors splitting and gives the same results in a much simpler way. Let us summarize the main features and the most interesting properties of this representation.

For any Feynman graph, the related Feynman amplitude is written as an integral of the Mellin-Barnes type: the CM integrand is a product of Γ-functions with linear arguments, and of linear powers of the external invariants and internal masses of the Feynman graph, while the integration path is the set of the imaginary axes. The linear polar varieties of the integrand form a lattice in the real hyperplane, depending only on the topology of the graph.

Thus the Cauchy theorem gives an easy determination of any asymptotic expansion, by translations of the integration path. This explicit study of the asymptotic behaviours, which motivated at first the CM representation, is explained in [1], some references and physical examples being quoted in [2].

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Another interesting property is that the renormalization takes a very simple form in the CM representation: it only results in translations of the integration path, without any change in the integrand. This was conjectured in [1], on the basis of simple examples. The main result of the present paper is to prove it, and to give explicitly the CM representation of the renormalized Feynman amplitudes. The fact that the integrand is unmodified allows to use the same method as in the convergent case, for the determination of asymptotic expansions. More generally, this fact is a nice property for all studies concerned with the dependence on the invariants (algebraic properties, differential equations etc...).

On the other hand, we think that the CM representation could be useful to the study of other topics: analytic renormalization [4], dimensional renormalization, Landau singularities. With regard to the dimensional regularization, we give some indications in our last section. The problem of the Landau singularities is not studied here: we use the euclidean metric throughout; the minkowskian case may then be reached by a Wick rotation, raising the question of the convergence at infinity of the CM integrals.

Finally we give the plan of our paper. In Sect. II, we recall the definition of the CM representation in the convergent case, and the notations. In Sect. III, we perform the renormalization and state the explicit CM representation of the renormalized Feynman amplitudes. Section IV gives further topics, namely the analytic continuation in dimension, and the extension of our representation to the case of particles with spin, and to the case of derivative couplings (for simplicity, we work in Sect. II and III with scalar amplitudes).

II. CM Representation of the Convergent Feynman Amplitudes

In this section, we mainly recall the results of [1] and the notation used in the following sections. Given any Feynman graph G with L_G independent loops, we label its l_G internal lines by i, the internal masses by m_i, its distinct one-trees (connected trees spanning all vertices of G) by j, its distinct two-trees (trees with two connected components, spanning all vertices of G) by k, the cut-invariant corresponding to any two-tree k by s_k (for different two-trees, the invariants s_k may coincide). Introducing u_{ij} (respectively u_{ik})$=0$ or 1 following the line i belongs or not to the one-tree j (respectively two-tree k), the Symanzik polynomials are:

$$U=\sum_j U_j=\sum_j \prod_i \alpha_i^{u_{ij}}$$

$$N=\sum_k N_k=\sum_k s_k \prod_i \alpha_i^{u_{ik}}$$

and the Feynman amplitude is given by the α representation:

$$F(s_k, m_i^2)=\int_0^\infty \prod_i d\alpha_i e^{-\sum_i \alpha_i m_i^2} U^{-D/2} e^{-N/U}, \tag{1}$$

where D is the space-time dimension. The Feynman integrand may then be expressed by a Mellin representation:

$$U^{-D/2}e^{-N/U}=I_{C_0}=\int_{C_0} \frac{\prod_j \Gamma(-x_j)}{\Gamma\left(-\sum_j x_j\right)} \prod_k s_k^{y_k} \Gamma(-y_k) \prod_i \alpha_i^{\varphi_i-1}, \tag{2}$$

where the complex variables x, y belong to the cell C_0:

$$C_0 = \left\{ x, y \,\middle|\, \sum_j x_j + \sum_k y_k = -\frac{D}{2};\ \mathrm{Re}\, x_j < 0\ \forall j;\ \mathrm{Re}\, y_k < 0\ \forall k \right\}. \tag{3}$$

The integration symbol means

$$\int_{-\infty}^{+\infty} \prod_j \frac{d\,\mathrm{Im}\, x_j}{2i\pi} \prod_k \frac{d\,\mathrm{Im}\, y_k}{2i\pi}\, 2i\pi\delta\left(\sum_j \mathrm{Im}\, x_j + \sum_k \mathrm{Im}\, y_k\right)$$

and

$$\varphi_i = \sum_j u_{ij} x_j + \sum_k u_{ik} y_k + 1. \tag{4}$$

The convergence of the Feynman integral is equivalent to the superficial convergence of every subgraph S of G:

$$\omega_S = l_S - \frac{D}{2} L_S > 0 \quad \forall S \subsetneqq G. \tag{5}$$

Theorem 1 in [1] says that this convergence is also equivalent to the following assertion:

$$C_0 \cap \Delta_G \neq \emptyset, \tag{6}$$

where

$$\Delta_G = \{ x, y \,|\, \mathrm{Re}\, \varphi_i > 0 \ \ \forall i \in G \}. \tag{7}$$

This is sufficient to exchange the x, y, and α integrations and to obtain the CM representation of the amplitude:

$$F(s_k, m_i^2) = \int_{C_0 \cap \Delta_G} \frac{\prod_j \Gamma(-x_j)}{\Gamma\left(-\sum_j x_j\right)} \prod_k s_k^{y_k} \Gamma(-y_k) \prod_i (m_i^2)^{-\varphi_i} \Gamma(\varphi_i). \tag{8}$$

Conversely let us assume there exists a divergent subgraph $S(\omega_S \leqq 0)$. We define:

$$\varphi_S = \sum_{i \in S} \varphi_i = \sum_j \left(\sum_{i \in S} u_{ij} \right) x_j + \sum_k \left(\sum_{i \in S} u_{ik} \right) y_k + l_S. \tag{9}$$

For getting any one-tree j or two-tree k of G, one must remove at least L_S lines from every subgraph S having L_S independent loops. Thus:

$$\sum_{i \in S} u_{ij} = L_S + a_{Sj}; \qquad \sum_{i \in S} u_{ik} = L_S + a_{Sk},$$

where a_{S_j}, a_{S_k} are non-negative integers.

Using $\sum_j x_j + \sum_k y_k = -\frac{D}{2}$ we find:

$$\varphi_S = \omega_S + \sum_j a_{Sj} x_j + \sum_k a_{Sk} y_k \tag{10}$$

which shows that $C_0 \cap \Delta_G$ is empty (ω_S, $\mathrm{Re}\, x_j$, $\mathrm{Re}\, y_k$ being not positive).

In [1] it was shown how the renormalization of the divergence corresponding to a subgraph S may be performed by translations of the integration path. But the generalization to the complete renormalization is not obvious and will be described in the following section.

III. CM Representation of the Renormalized Divergent Feynman Amplitudes

We rewrite the Feynman integrand defined in (2) as:

$$I_{C_0} = \int_{C_0} \prod_{\nu=1}^{N} \Gamma(-z_\nu) \prod_i \alpha_i^{\varphi_i - 1} f(z_\nu), \tag{11}$$

where the z_ν variables stand for the x_j and y_k variables,

$$f(z_\nu) = \frac{1}{\Gamma\left(-\sum_j x_j\right)} \prod_k s_k^{y_k}$$

is analytic, and

$$C_0 = \left\{ z \,\middle|\, \sum_{\nu=1}^{N} z_\nu = -\frac{D}{2};\ \mathrm{Re}\, z_\nu < 0\ \forall \nu \right\}. \tag{12}$$

We use the renormalization defined by the $\mathscr{R}$ operation of Bergère and Lam [5]:

$$\mathscr{R} I_{C_0} = \left[\prod_{S \subsetneq G} (1 - \mathscr{T}_S) \right] I_{C_0}. \tag{13}$$

Let us recall how this $\mathscr{R}$ operation applies to any function $g(\alpha)$. For each subgraph S, the α_i variables, $i \in S$, are scaled by ϱ_S, and one writes the generalized expansion in ϱ_S:

$$g(\alpha; \varrho_S) = \sum_{p=p_{\min}}^{+\infty} \varrho_S^p g_p(\alpha).$$

Then by definition the $\mathscr{T}_S$ operator only retains the beginning of this expansion, namely those terms with $p \leqq -l_S$, and one finally takes $\varrho_S = 1$. If $p_{\min} > -l_S$ (superficial convergence of the subgraph S), one has: $\mathscr{T}_S g = 0$.

In order to obtain the Taylor expansions of I_{C_0}, we will use the method described in [1]. By translating the integration path of (11), the various terms of the Taylor expansions are given by the residues of the simple poles of the functions $\Gamma(-z_\nu)$. And these residues will be written as differences of integrals I_C in various regions, or "cells" C, in the space of the z variables.

Definition 1. A cell C is defined by a set of non-negative integers $n_\nu(C)$ (one of them at least being zero):

$$C = \left\{ z \,\middle|\, \begin{array}{ll} \displaystyle\sum_{\nu=1}^{N} z_\nu = -\frac{D}{2}; & \mathrm{Re}\, z_\nu < 0 \quad \forall \nu \text{ with } n_\nu = 0 \\ n_\nu - 1 < \mathrm{Re}\, z_\nu < n_\nu & \forall \nu \text{ with } n_\nu \geqq 1 \end{array} \right\}. \tag{14}$$

Definition 2. Let A be the set of values taken by the index ν ($|A| = N$ is the number of one-trees plus the number of two-trees of the graph). For any strict subset E of A

($|E|<N$), we define a partial ordering of the cells by:

$$C_2 \overset{E}{\geqq} C_1 \Leftrightarrow \begin{cases} n_v(C_2) \geqq n_v(C_1) & \forall v \in E \\ n_v(C_2) = n_v(C_1) & \forall v \notin E. \end{cases} \tag{15}$$

Definition 3. Given two cells C_1, C_2 with closures $\bar{C}_1$, $\bar{C}_2$, we define their relative coefficient $\varepsilon_{C_1 C_2}$ by:

i) if $\exists E$, $C_2 \overset{E}{\geqq} C_1$; $\nexists F \subset E$, $C_2 \overset{F}{\geqq} C_1$ and $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$ i.e. if $n_v(C_2) = n_v(C_1) + \theta_{vE}$ where $\theta_{vE} = \begin{vmatrix} 1 & \forall v \in E \\ 0 & \forall v \notin E \end{vmatrix}$ then we put

$$\varepsilon_{C_1 C_2} = (-1)^{|E|}. \tag{16}$$

For the particular case $E = \emptyset$, we have $\varepsilon_{CC} = 1 \; \forall C$.

ii) In every other case, we put $\varepsilon_{C_1 C_2} = 0$.

These coefficients satisfy:

$$\sum_{C \overset{E}{\geqq} C_1} \varepsilon_{C_1 C} = 0 \quad \forall C_1, \forall E.$$

From this identity one can easily deduce the following lemmas:

Lemma 1. *Given E and $C_1 \overset{E}{\geqq} C_0$, $C_2 \overset{E}{\geqq} C_0$, where C_0 is defined in (12), we have:*

$$\sum_{C_1 \overset{E'}{\leqq} C \overset{E'}{\leqq} C_2} \varepsilon_{CC_2} = \delta_{C_1 C_2} \quad \forall E' \subseteq E. \tag{17}$$

Lemma 2. *More generally, given $C_1 \overset{E}{\geqq} C_0$, $C_2 \overset{E}{\geqq} C_0$, given a subset F of E and a set $\mathcal{C}$ of cells, such that*

$$C \in \mathcal{C} \Rightarrow C' \in \mathcal{C} \quad \forall C' \overset{E-F}{\geqq} C \quad \text{and} \quad \forall C' \overset{E-F}{\leqq} C$$

then we have:

$$\sum_{\substack{C \in \mathcal{C} \\ C_1 \overset{E'}{\leqq} C \overset{E'}{\leqq} C_2}} \varepsilon_{CC_2} = \sum_{\substack{C \in \mathcal{C} \\ C_1 \overset{F}{\leqq} C \overset{F}{\leqq} C_2}} \varepsilon_{CC_2} \quad \forall E' \text{ with } F \subsetneq E' \subsetneq E. \tag{18}$$

The action of the Taylor operators in the renormalization procedure may now be described in the following way; to each subgraph S corresponds the linear form already given in (10):

$$\varphi_S = \omega_S + \sum_v a_{Sv} z_v, \tag{19}$$

where

$$a_{Sv} = \sum_{i \in S} u_{iv} - L_S \geqq 0 \quad \forall v.$$

Moreover $\{v | a_{Sv} = 0\} \neq \emptyset$ since there are one-trees of G, whose restrictions to S are one-trees of S. Let us call E_S the strict subset of A

$$E_S = \{v | a_{Sv} > 0\} \tag{20}$$

and $\overset{S}{\geqq}$ the corresponding partial ordering $\overset{E_S}{\geqq}$.

From Definition 3, for each cell $C \overset{E}{\geqq} C_0$, with $E_S \subseteq E \subset A$, $\sum_{C' \overset{S}{\geqq} C} \varepsilon_{CC'} I_{C'}$, is the partial residue of I_C at

$$z_v = n_v(C) \quad \forall v \in E_S .$$

Thus if the α_i's, $i \in S$, are scaled by ϱ_S, the Taylor expansion of $I_C(\varrho_S)$ is

$$\sum_{C' \overset{S}{\geqq} C} \Big(\sum_{C'' \overset{S}{\geqq} C'} \varepsilon_{C'C''} I_{C''} \Big) \varrho_S^{\varphi_S(n_v(C)) - l_S} . \tag{21}$$

This expansion may be obtained by writing from Lemma 1 the identity

$$I_C = \sum_{C''} \Big(\sum_{C \overset{S}{\geqq} C' \overset{S}{\geqq} C''} \varepsilon_{C'C''} \Big) I_{C''} \tag{22}$$

and interchanging the order of summations.

By definition of the Taylor operations, we get:

$$\mathscr{T}_S I_C = \sum_{\substack{C' \overset{S}{\geqq} C \\ \sup_{C'} \varphi_S \leqq 0}} \sum_{C'' \overset{S}{\geqq} C'} \varepsilon_{C'C''} I_{C''} . \tag{23}$$

This may be generalized to every nested set of subgraphs:

$$\mathscr{N} = \{S_1, \ldots, S_n\} , \quad S_1 \subset S_2 \subset \ldots \subset S_n .$$

Indeed, the Feynman integrand is FINE [2], i.e. it has a simultaneous Taylor expansion in the scaling parameters ϱ_S of every nest $\mathscr{N}$. To be explicit, if $\forall S \in \mathscr{N}$, $\alpha_i \to \varrho_S \alpha_i$, $\forall i \in S$, then $U(\varrho_S) = U_{\mathscr{N}} \prod_{S \in \mathscr{N}} \varrho_S^{L_S} +$ higher order terms, with $U_{\mathscr{N}} = U_{S_1} U_{S_2/S_1} \ldots U_{G/S_n}$, the S_p/S_{p-1} being the usual reduced graphs.

Let us call $E_{\mathscr{N}}$ the union $\bigcup_{S \in \mathscr{N}} E_S$ (and $\overset{\mathscr{N}}{\geqq}$ the corresponding ordering $\overset{E_{\mathscr{N}}}{\geqq}$). $E_{\mathscr{N}}$ is a strict subset of A since for the monomials j of $U_{\mathscr{N}}$

$$a_{Sj} = \sum_{i \in S} u_{ij} - L_S = 0 \quad \forall S \in \mathscr{N} \Rightarrow j \notin E_{\mathscr{N}}$$

and the simultaneous Taylor expansion of I_{C_0} is

$$\sum_{C \overset{\mathscr{N}}{\geqq} C_0} \Big(\sum_{C' \overset{\mathscr{N}}{\geqq} C} \varepsilon_{CC'} I_{C'} \Big) \prod_{S \in \mathscr{N}} \varrho_S^{\varphi_S(n_v(C)) - l_S} . \tag{24}$$

From the definition of the Taylor operations, we retain only those terms for which

$$\varphi_S(n_v(C)) = \sup_C \varphi_S \leqq 0 \quad \forall S \in \mathscr{N} .$$

By interchanging the order of summations, we may then state:

Lemma 3.

$$\Big(\prod_{S \in \mathscr{N}} \mathscr{T}_S \Big) I_{C_0} = \sum_C \mu_{C,\mathscr{N}} I_C , \tag{25}$$

$$\mu_{C,\mathscr{N}} = \sum_{C' \in \mathscr{E}_{\mathscr{N}}(C)} \varepsilon_{C'C} , \tag{26}$$

$$\mathscr{E}_{\mathscr{N}}(C) = \Big\{ C' \Big| C_0 \overset{\mathscr{N}}{\leqq} C' \overset{\mathscr{N}}{\leqq} C ; \sup_{C'} \varphi_S \leqq 0 \ \ \forall S \in \mathscr{N} \Big\} . \tag{27}$$

Remarks. In contrast with the formal infinite series (21) or (24), there is only a finite number of cells C with non-vanishing multiplicity $\mu_{C\mathcal{N}}$ in (25), due to the conditions in (27). From $\varepsilon_{C'C}=0$ if $\bar{C}'\cap\bar{C}=\emptyset$, and $C'\overset{\mathcal{N}}{\leqq}C$, we have indeed:

$$\mu_{C\mathcal{N}}=0 \quad \text{if} \quad \exists S\in\mathcal{N} \quad \text{with} \quad \inf_C \varphi_S>0. \tag{28}$$

Moreover, apart from C_0, the cells with a non-vanishing multiplicity do not lie inside the domain $\{\varphi_S\leqq 0 \ \forall S\in\mathcal{N}\}$, but only on its border, since from (27) and Lemma 1 we have:

$$\mu_{C\mathcal{N}}=\delta_{C_0C} \quad \text{if} \quad \sup_C \varphi_S\leqq 0 \quad \forall S\in\mathcal{N}. \tag{29}$$

We can now give the renormalized integrand, since the $\mathscr{R}$ operation given in (13) can also be written [5]:

$$\mathscr{R}=\sum_{\mathcal{N}}\prod_{S\in\mathcal{N}}(-\mathscr{T}_S), \tag{30}$$

where the summation is performed over all the nests, including the empty nest $\mathcal{N}=\emptyset$, for which $\prod_{S\in\emptyset}(-\mathscr{T}_S)=1$. So we state:

Theorem 1.

$$\mathscr{R}I_{C_0}=\sum_C \mu_C I_C, \tag{31}$$

where

$$\mu_C=\sum_{\mathcal{N}}(-1)^{|\mathcal{N}|}\mu_{C\mathcal{N}} \tag{32}$$

and $\mu_{C\mathcal{N}}$ is defined in Lemma 3.

Before proceeding to the second step, that is the α integrations, we must summarize some of the results of Bergère and Lam [5].

Let $\mathcal{M}$ be a fixed maximal nest of subgraphs R_m of G:

$$\mathcal{M}=\{R_1,R_2,\ldots,R_{l_G}\}, \qquad R_{m+1}=R_m\cup\{i_{m+1}\}.$$

For a given $\mathcal{M}$, the set of all nests $\mathcal{N}$ can be split into equivalence classes. Each class Γ is characterized by its maximal nest $\mathcal{M}(\Gamma)$ which is the union of two disjoint subnests $\mathscr{H}(\Gamma)$ and $\mathscr{K}(\Gamma)$ such that:

i) $\mathcal{N}\in\Gamma\Leftrightarrow\mathscr{K}(\Gamma)\subseteq\mathcal{N}\subseteq\mathscr{K}(\Gamma)\cup\mathscr{H}(\Gamma)=\mathcal{M}(\Gamma)$.

Therefore:

$$\sum_{\mathcal{N}\in\Gamma}\prod_{S\in\mathcal{N}}(-\mathscr{T}_S)=\prod_{K\in\mathscr{K}(\Gamma)}(-\mathscr{T}_K)\prod_{H\in\mathscr{H}(\Gamma)}(1-\mathscr{T}_H). \tag{33}$$

ii) To each $R_m\in\mathcal{M}$ and each class Γ correspond subsets $\{H_{mj}\}$ of $\mathscr{H}(\Gamma)$, $\{K_{mj}\}$ of $\mathscr{K}(\Gamma)$ satisfying

$$K_{mj}\subset H_{mj} \qquad \forall j$$

$$(H_{mj}-K_{mj})\cap(H_{mj'}-K_{mj'})=\emptyset \qquad \forall j\neq j'$$

$$R_m=\bigcup_j(H_{mj}-K_{mj})$$

which means with our notations:

$$\varphi_{R_m} = \sum_j \varphi_{H_{mj}} - \sum_j \varphi_{K_{mj}}. \tag{34}$$

Apart from these results, we need the following lemma:

Lemma 4. *Given two disjoint nests $\mathscr{A}, \mathscr{B}$, the union of which is a nest $\mathscr{N}$, we have:*

$$\left[\prod_{T\in\mathscr{B}} \mathscr{T}_T \prod_{S\in\mathscr{A}} (1-\mathscr{T}_S)\right] I_{C_0} = \sum_C \mu_{C\mathscr{A}\mathscr{B}} I_C, \tag{35}$$

where

$$\mu_{C\mathscr{A}\mathscr{B}} = \sum_{C'\in\mathscr{E}_{\mathscr{A}\mathscr{B}}(C)} \varepsilon_{C'C}, \tag{36}$$

$$\mathscr{E}_{\mathscr{A}\mathscr{B}}(C) = \left\{ C' \left| \begin{array}{c} C_0 \overset{\mathscr{N}}{\subseteq} C' \overset{\mathscr{N}}{\subseteq} C;\ \sup_{C'} \varphi_T \leqq 0 \ \ \forall T\in\mathscr{B}; \\ \sup_{C'} \varphi_S > 0 \ \ \forall S\in\mathscr{A} \end{array} \right. \right\}. \tag{37}$$

Proof. The proof is by recurrence on the number of subgraphs S in $\mathscr{A}$. For $\mathscr{A}=\emptyset$, Lemma 4 is nothing but Lemma 3. Let us assume that Lemma 4 is true for $\mathscr{A}' = \mathscr{A} - \{S_1\}$ and any second subnest, in particular $\mathscr{B}$ and $\mathscr{B}' = \mathscr{B}\cup\{S_1\}$. Then:

$$\begin{aligned} &\prod_{T\in\mathscr{B}} \mathscr{T}_T \prod_{S\in\mathscr{A}} (1-\mathscr{T}_S) I_{C_0} \\ &\quad = \prod_{T\in\mathscr{B}} \mathscr{T}_T \prod_{S\in\mathscr{A}'} (1-\mathscr{T}_S) I_{C_0} - \prod_{T\in\mathscr{B}'} \mathscr{T}_T \prod_{S\in\mathscr{A}'} (1-\mathscr{T}_S) I_{C_0} \\ &\quad = \sum_C \lambda_{C\mathscr{A}\mathscr{B}} I_C \end{aligned} \tag{38}$$

with

$$\lambda_{C\mathscr{A}\mathscr{B}} = \mu_{C\mathscr{A}'\mathscr{B}} - \mu_{C\mathscr{A}'\mathscr{B}'}. \tag{39}$$

But from Lemma 2

$$\mu_{C\mathscr{A}'\mathscr{B}} = \sum_{C'\in\mathscr{E}'_{\mathscr{A}'\mathscr{B}}(C)} \varepsilon_{C'C} = \sum_{C'\in\mathscr{E}'_{\mathscr{A}'\mathscr{B}}(C)} \varepsilon_{C'C}, \tag{40}$$

where

$$\mathscr{E}'_{\mathscr{A}'\mathscr{B}}(C) = \left\{ C' \left| \begin{array}{c} C_0 \overset{\mathscr{N}}{\subseteq} C' \overset{\mathscr{N}}{\subseteq} C;\ \sup_{C'} \varphi_T \leqq 0 \ \ \forall T\in\mathscr{B}; \\ \sup_{C'} \varphi_S > 0 \ \ \forall S\in\mathscr{A}' \end{array} \right. \right\}.$$

By replacing (40) into (39), one sees $\lambda_{C\mathscr{A}\mathscr{B}} = \mu_{C\mathscr{A}\mathscr{B}}$, which achieves the proof of Lemma 4.

We may now use (30), (31), (33), and Lemma 4 to write:

Lemma 5.

$$\mu_C = \sum_\Gamma \mu_{C\Gamma} \tag{41}$$

with

$$\mu_{C\Gamma} = \sum_{\mathcal{N} \in \Gamma} (-1)^{|\mathcal{N}|} \mu_{C\mathcal{N}} = (-1)^{|\mathcal{K}(\Gamma)|} \sum_{C' \in \mathscr{E}_\Gamma(C)} \varepsilon_{C'C}, \tag{42}$$

$$\mathscr{E}_\Gamma(C) = \left\{ C' \middle| C_0 \overset{\mathscr{U}(\Gamma)}{\leqq} C' \overset{\mathscr{U}(\Gamma)}{\leqq} C;\ \sup_{C'} \varphi_K \leqq 0\ \forall K \in \mathscr{K}(\Gamma);\ \sup_{C'} \varphi_H > 0\ \forall H \in \mathscr{H}(\Gamma) \right\}. \tag{43}$$

Remarks. For each Γ, there is only a finite number of cells C such that $\mu_{C\Gamma} \neq 0$, since $\mu_{C\Gamma}$ is a finite combination of $\mu_{C\mathcal{N}}$'s. Moreover many of the $\mu_{C\Gamma}$'s vanish, since from (43):

$$\mu_{C\Gamma} = 0 \quad \text{if} \quad \exists K \in \mathscr{K}(\Gamma) \quad \text{with} \quad \inf_C \varphi_K > 0, \tag{44}$$

$$\mu_{C\Gamma} = 0 \quad \text{if} \quad \exists H \in \mathscr{H}(\Gamma) \quad \text{with} \quad \sup_C \varphi_H \leqq 0, \tag{45}$$

$$\mu_{C\Gamma} = (-1)^{|\mathcal{K}(\Gamma)|} \delta_{C_0 C} \quad \text{if} \quad \inf_C \varphi_H > 0\ \forall H \in \mathscr{H}(\Gamma) \quad \text{and} \quad \sup_C \varphi_K \leqq 0\ \forall K \in \mathscr{K}(\Gamma). \tag{46}$$

The related vanishing of many μ_C's also appears in the following lemma:

Lemma 6. *For each cell C with $\mu_C \neq 0$, we have:*

$$C \cap \Delta_{\mathcal{M}} \neq \emptyset, \tag{47}$$

where

$$\Delta_{\mathcal{M}} = \{z | \operatorname{Re} \varphi_{R_m} > 0\ \forall R_m \in \mathcal{M}\}. \tag{48}$$

Proof. If $\mu_C \neq 0$, $\exists \Gamma$ with $\mu_{C\Gamma} \neq 0$. Hence

$$\exists C' \overset{\mathcal{M}(\Gamma)}{\leqq} C \quad \text{with} \quad \begin{cases} \varepsilon_{C'C} \neq 0 \\ \sup_{C'} \varphi_H > 0 \quad \forall H \in \mathscr{H}(\Gamma) \\ \sup_{C'} \varphi_K \leqq 0 \quad \forall K \in \mathscr{K}(\Gamma). \end{cases} \tag{49}$$

Let $\sigma(C')$ be the edge of C': $z_v = n_v(C')\ \forall v \in E_{\mathcal{M}(\Gamma)}$. From the positivity of the coefficients a_{Sv} in φ_S, $\forall S \in \mathcal{M}(\Gamma)$:

$$\varphi_H(\sigma(C')) = \sup_{C'} \varphi_H > 0$$

$$\varphi_K(\sigma(C')) = \sup_{C'} \varphi_K \leqq 0$$

whence from (34):

$$\varphi_{R_m}(\sigma(C')) > 0 \quad \forall R_m \in \mathcal{M}. \tag{50}$$

But $C' \overset{\mathcal{M}(\Gamma)}{\leqq} C$ and $\bar{C}' \cap \bar{C} \neq \emptyset$ imply $\sigma(C') \in \bar{C}$.

Thus in the neighbourhood of $\sigma(C')$ there exist points in C where the φ_{R_m} are simultaneously positive, which proves Lemma 6.

Now the result expressed by Lemma 6 is true whatever is the fixed maximal nest $\mathcal{M}=\{R_m\}$ originally choosen. Therefore we shall be able to perform the α integrations with the use of only one further lemma:

Lemma 7. *Given a cell C, if for each maximal nest $\mathcal{M}$, $C\cap\Delta_{\mathcal{M}}$ is not empty, then:*

$$\Delta_C=C\cap\Delta_G\neq\emptyset, \tag{51}$$

where

$$\Delta_G=\{z|\operatorname{Re}\varphi_i>0\ \forall i\in G\}. \tag{52}$$

Proof. In the space of the Rez's, any cell C is defined by a set of linear boundary conditions, which we write $\psi_\beta>0\ \forall\beta$. If in C the φ_i's cannot be simultaneously positive, there must exist non-negative real numbers a_0, a_i, a_β, the a_i's being not all vanishing, with

$$\sum_i a_i\varphi_i+\sum_\beta a_\beta\psi_\beta+a_0\equiv 0. \tag{53}$$

By ordering the a_i's: $a_{i_m}\geqq a_{i_{m-1}}\ \forall m>1$ and putting: $b_m=a_{i_m}-a_{i_{m-1}}$, $\forall m>1$; $b_1=a_{i_1}$ we get a set of non-negative integers b_m, not all vanishing, with:

$$\sum_m b_m\varphi_{R_m}+\sum_\beta a_\beta\psi_\beta+a_0\equiv 0, \tag{54}$$

where

$$R_m=\{i_m,i_{m+1},\ldots,i_{l_G}\}. \tag{55}$$

Then there would exist a maximal nest, the φ_{R_m} of which cannot be simultaneously positive in C, which would contradict the hypothesis.

By Lemma 7, the α integrations may now be performed for $z\in\Delta_C$ and we set up the results of this section in the following theorem:

Theorem 2. *The renormalized Feynman amplitude corresponding to a graph G is given by the* CM *representation:*

$$F_{\mathcal{R}}(s_k,m_i^2)=\sum_C\mu_C\int_{\Delta_C}\frac{\prod_j\Gamma(-x_j)}{\Gamma\left(-\sum_j x_j\right)}\prod_k s_k^{y_k}\Gamma(-y_k)\prod_i(m_i^2)^{-\varphi_i}\Gamma(\varphi_i), \tag{56}$$

where the multiplicities μ_C are defined in Theorem 1 and Lemma 3, and $\mu_C=0$ if $\Delta_C=C\cap\Delta_G$ is empty.

Remarks. We used in this section the $\mathcal{R}$ operation of Bergère and Lam, and the same machinery of equivalence classes of nests [5]. But it can be noticed that this section gives a proof independent of theirs for the convergence of the renormalized amplitudes. We find $F_{\mathcal{R}}$ as a sum of integrals. However, in contrast with the usual renormalization schemes, the integrands are unmodified, and each of these integrals is separately convergent. The set of cells with non-vanishing multiplicity is a finite set, and these cells lie on the border of Δ_G:

$$\mu_C\neq 0\Rightarrow\begin{cases}\exists z\in C & \text{with}\quad \operatorname{Re}\varphi_i(z)>0\quad\forall i\\ \exists i & \text{with}\quad \operatorname*{Inf}_C\varphi_i\leqq 0.\end{cases} \tag{57}$$

Translating the integration path farther inside Δ_G would correspond to performing oversubtractions, which could be expressed in our formalism by the simple replacements:

$$\begin{aligned} &\varphi_S \to \varphi'_S = \varphi_S - d_S \\ &\Delta_G \to \Delta'_G = \{z | \mathrm{Re}\,\varphi'_S > 0 \ \ \forall S \subsetneq G\}. \end{aligned} \tag{58}$$

IV. Additional Topics

1. Dimensional Regularization

The CM representation can likely be used to the explicit study of the dimensionally renormalized amplitudes. We do not attempt here to perform the dimensional renormalization, but only give indications on the dimensional regularization. Starting with a convergent Feynman amplitude (for a sufficiently low value of the space-time dimension), let us show how we can reach the following results:

– the amplitude is a meromorphic function of the dimension.

– its dimensional analytic continuation has a CM representation with the same unchanged CM integrand, and again translations of the integration path.

The first of these results is already known (see e.g. [6]), but can be directly proved from the CM representation, in exactly the same way as was determined any asymptotic expansion in [1]. Let $F(D)$ be a Feynman amplitude, convergent for $\mathrm{Re}\,D < D_0$: it admits a CM representation given by Eq. (8), which we rewrite as:

$$F(D) = \int\limits_{\gamma_0 \cap P(D)} \prod_r \Gamma(\varphi_r) J(z), \tag{59}$$

where the z_ν variables stand for the x_j and y_k variables as in (11), the linear forms φ_r stand for $-x_j$, $-y_k$, and φ_i,

$$J(z) = \frac{1}{\Gamma\left(-\sum\limits_j x_j\right)} \prod_k s_k^{y_k} \prod_i (m_i^2)^{-\varphi_i}$$

is analytic,

$$\gamma_0 = \{z | \mathrm{Re}\,\varphi_r > 0 \ \ \forall r\}.$$

$P(D)$ is the hyperplane $\sum\limits_\nu z_\nu = -\dfrac{D}{2}$.

The lattice of the polar varieties $\{\varphi_r = -n_r\}$, where the n_r's are non-negative integers, gives the boundaries of a set of cells

$$\gamma = \{z | \mathrm{Re}\,\varphi_r > 0 \text{ for } n_r = 0;\ -n_r < \mathrm{Re}\,\varphi_r < -n_r + 1 \text{ for } n_r \geqq 1\}.$$

For a given cell γ, we define D_m and D_M by:

$$-\frac{D_m(\gamma)}{2} = \sup_\gamma \left(\sum_\nu \mathrm{Re}\, z_\nu\right); \qquad -\frac{D_M(\gamma)}{2} = \mathrm{Inf}_\gamma \left(\sum_\nu \mathrm{Re}\, z_\nu\right). \tag{60}$$

Then $F_\gamma(D) = \int\limits_{\gamma \cap P(D)} \prod\limits_r \Gamma(\varphi_r) J(z)$ defines an analytic function of D in the strip $D_m < \mathrm{Re}\,D < D_M$. In the same way as in Sect. III of [1], we use the identities

expressing that $\sum_v z_v + D_M(\gamma)/2$ belongs to the convex spaces generated by some subsets of the boundary forms of γ. We obtain:

$$\left.\frac{1}{\prod_r(\varphi_r+n_r)}\right|_{P(D)} = \sum_E \frac{d_E}{(D-D_M)^{qE-1}} \frac{1}{\prod_{r\in E}(\varphi_r+n_r)} \tag{61}$$

where the sets $\{\varphi_r, r\in E\}$ no longer generate $D-D_M$, which implies that each term in (61) may be analytically continued to higher values of D.

Thus, starting from γ_0, one can explicitly extract the various poles in dimension, and iteratively translate the integration path to other cells. This shows the meromorphy in dimension; the poles in D belong to the discrete set of rational numbers corresponding to the values of $\sum_v z_v$ at the vertices of the lattice $\{\varphi_r=-n_r\}$.

An alternative way of working is the following: we rewrite (59) as a difference of integrals over all the $\mathrm{Im} z_v$'s $\left(\text{no longer linked by the constraint } \sum_v z_v = -\frac{D}{2}\right)$:

$$F(D) = \int_{\gamma_0\cap B_+(D)} \frac{\prod_r \Gamma(\varphi_r)}{\varphi_0} J(z) - \int_{\gamma_0\cap B_-(D)} \frac{\prod_r \Gamma(\varphi_r)}{\varphi_0} J(z), \tag{62}$$

where

$$B_+(D) = \{z \mid 0 < \mathrm{Re}\,\varphi_0 < \varepsilon\}$$

$$B_-(D) = \{z \mid -\varepsilon < \mathrm{Re}\,\varphi_0 < 0\}$$

$$\varphi_0 = \sum_v z_v + \frac{D}{2}$$

and ε is a small positive real number.

In these integrals, we translate the integration path by increasing the dimension from D to D' and crossing the various singularities of the lattice $\{\varphi_r=-n_r\}$.

Thus we obtain:

$$\int_{\gamma_0\cap B_\pm(D)} \frac{\prod_r \Gamma(\varphi_r)}{\varphi_0(D)} J(z) = \sum_\gamma \lambda_\gamma \int_{\gamma\cap B_\pm(D')} \frac{\prod_r \Gamma(\varphi_r)}{\varphi_0(D')} J(z) + \text{residues}. \tag{63}$$

In contrast with Sect. III (where we considered only the sub-lattice $\{-z_v=-n_v\}$), we generally get residues of multiple poles. In any case these residues disappear in the difference (62) if D' is such that the domain $\{z \mid -\varepsilon < \mathrm{Re}\,\varphi_0(D') < \varepsilon\}$ does not contain any vertex of the lattice. This condition corresponds to fix $\mathrm{Re}\,D'$ between two consecutive poles in dimension. Finally we can come back to the original form of the integral:

$$F(D') = \sum_\gamma \lambda_\gamma \int_{\gamma\cap P(D')} \prod_r \Gamma(\varphi_r)\, J(z). \tag{64}$$

The result (64) is similar to that one of our Sect. III: provided that the hyperplane $P(D')$ does not across the vertices of the lattice $\{\varphi_r = -n_r\}$ [that is, for $\mathrm{Re}D'$ distinct from the poles of $F(D)$], then the continued amplitude has a CM representation with unchanged integrand and translated integration path. Of course the multiplicities of the cells in (64) and (56) differ.

To obtain the same result, we could also start from the expression of the regularized amplitude given in [6]. For $\mathrm{Re}D$ between two consecutive poles:

$$F(D) = \lim_{\mu_i^2 \to 0} \int_0^\infty \prod_i d\alpha_i e^{-\sum_i \alpha_i \mu_i^2} \mathscr{R}\left[e^{-\sum_i m_i^2 \alpha_i} U^{-D/2} e^{-N/U} \right]. \tag{65}$$

The bracket in (65) has again a CM representation, with additional variables for the new factor:

$$e^{-\sum_i \alpha_i m_i^2} = \int_{\mathrm{Re}\, t_i < 0} \prod_i \left[\frac{d\,\mathrm{Im}\, t_i}{2i\pi} \Gamma(-t_i)(m_i^2)^{t_i} \alpha_i^{t_i} \right]. \tag{66}$$

Then the $\mathscr{R}$ operation acts in the same way as in Sect. III, with the replacement: $\varphi_i \to \psi_i = \varphi_i + t_i$.

Finally the limit $\mu_i^2 \to 0$ is easily taken, as explained at the end of Sect. III of [1], giving back (64).

2. *The Case of Non-Scalar Amplitudes*

In this paragraph, we want to generalize our CM representation to the case of particles with spin, and/or to derivative couplings. In the momentum representation, the Feynman amplitude corresponding to a given graph becomes:

$$F(p_v, m_i^2) = \int \prod_{a=1}^{L} d^D k_a \frac{\mathscr{P}(q_i)}{\prod_i (q_i^2 + m_i^2)}, \tag{67}$$

where the p_v's are the external momenta, the k_a's are the loop variables, the q_i's are the internal momenta, and $\mathscr{P}$ is a polynomial in the momenta, coming from the derivative couplings and/or from the numerators of the propagators. We take an α representation by:

$$\frac{q_i^\mu q_i^\nu \dots q_i^\sigma}{q_i^2 + m_i^2} = \int_0^\infty d\alpha_i \left\{ \left(-\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i^\mu} \right) \left(-\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i^\nu} \right) \dots \left(-\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i^\sigma} \right) e^{-\alpha_i (q_i^2 + m_i^2 + q_i \cdot \zeta_i)} \right\}_{\zeta_i = 0} \tag{68}$$

which gives the following α representation of the amplitude:

$$F(p_v, m_i^2) = \int_0^\infty \prod_i d\alpha_i \left\{ \mathscr{P}\left(-\frac{1}{\alpha_i} \frac{\partial}{\partial \zeta_i} \right) e^{-\sum_i \alpha_i \bar{m}_i^2} U^{-\frac{D}{2}} e^{-\frac{N(\bar{s}_k)}{U}} \right\}_{\zeta_i = 0}. \tag{69}$$

Here $\bar{m}_i^2 = m_i^2 - \dfrac{\zeta_i^2}{4}$.

$\bar{s}_k$ is the cut-invariant of the two-tree k, with the external momenta p_v replaced by $\bar{p}_v = p_v + \varepsilon_{vi} \dfrac{\zeta_i}{2}$, ε_{vi} being the usual incidence matrix.

Then we take a CM representation of (69) in the same way as before. For each monomial $\prod_i \left(-\frac{1}{\alpha_i}\frac{\partial}{\partial\zeta_i}\right)^{d_i}$ of $\mathscr{P}$ we obtain:

$$\int_{\Delta}\left\{\prod_i\left(-\frac{\partial}{\partial\zeta_i}\right)^{d_i}\frac{\prod_j\Gamma(-x_j)}{\Gamma\left(-\sum_j x_j\right)}\prod_k \bar{s}_k^{y_k}\Gamma(-y_k)\prod_i(\bar{m}_i^2)^{-\varphi_i-d_i}\Gamma(\varphi_i-d_i)\right\}_{\zeta_i=0}. \tag{70}$$

In (70), the action of the differential operators $\frac{\partial}{\partial\zeta_i}$ results in translations of the linear forms $-y_k$ or φ_i-d_i, and factorization of tensors built from the external momenta. The main difference with the scalar case is the complicated list of cells which enter into the new integration domains Δ, depending on the various monomials of $\mathscr{P}$. Otherwise the renormalization will follow the same lines as for the scalar amplitudes.

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TROISIEME PARTIE

Méthodes graphiques et inégalités de corrélation.

"Les Oulhamrs fuyaient dans la nuit épouvantable. Fous de souffrance et de fatigue, tout leur semblait vain devant la calamité suprême: le Feu était mort."

J. H. Rosny Aîné.

$$<\sigma_\alpha\sigma_\gamma> \leq \sum_{\beta\in B} <\sigma_\alpha\sigma_\beta>_A <\sigma_\beta\sigma_\gamma> \qquad \text{(III.1.2)}$$

où la restriction $<\sigma_\alpha\sigma_\beta>_A$ indique qu'on calcule la valeur moyenne dans le système A seulement, A étant la partie connexe du réseau Λ-B qui contient le site α. Par la seconde inégalité de Griffiths (III.1.2) implique (III.1.1). La démonstration de Lieb reposait sur une inégalité de type gaussien qui se déduit facilement des inégalités de Newman [62] dans le cas du modèle d'Ising. Considérons une famille complète orthonormale $\{\phi^n\}$ de fonctions dans $L^2(d_\mu)$ où d_μ est la mesure associée à un site du réseau. Pour un multi-index $n = (n_1, n_2 \ldots)$ sur $B = (b_1, b_2 \ldots)$ on définit: $\phi_B^n = \phi_{b_1}^{n_1} \phi_{b_2}^{n_2} \ldots$.
B désigne comme précédemment un ensemble de spins séparant A de C. L'inégalité utilisée par Lieb affirme alors l'existence d'une fonction F(n) telle que:

$$<\sigma_A \Phi_B^m>_A \leq \sum_n F(n) <\sigma_A \phi_B^n>_A <\bar{\phi}_B^n \phi_B^m>_A \qquad \text{(III.1.3)}$$

pour tout m tel que $<\sigma_C \phi_B^m>_C > 0$; la sommation se fait sur les multi-index n.

Notre contribution a consisté à établir l'inégalité de type gaussien (III.1.3) et donc l'inégalité (III.1.2) de Lieb pour le modèle à deux composantes des rotateurs plans. Pour cela nous avons démontré une conjecture portant sur des graphes orientés proposée par Lieb [54].

III.1, Introduction

L'article qui va suivre s'insère dans une série de résultats obtenus en 1980 par Aizenman , Lieb et Simon et publiés dans le même numéro de la revue Communications in Mathematical Physics [52-54]. Comme l'indique B. Simon, le point de départ était un résultat de Dobrushin [65], selon lequel pour certains modèles une décroissance polynomiale suffisamment forte des fonctions de corrélation à grande distance entraîne automatiquement une décroissance exponentielle. Simon découvrit que pour le modèle d'Ising ce phénomène est la conséquence d'une inégalité de corrélation très simple:

$$\langle\sigma_\alpha\sigma_\gamma\rangle \leq \sum_{\beta\in B} \langle\sigma_\alpha\sigma_\beta\rangle\langle\sigma_\beta\sigma_\gamma\rangle \qquad \text{(III.1.1)}$$

lorsque l'ensemble des sites de B sépare le réseau Λ en deux composantes connexes, l'une, A, contenant le site α, l'autre, C, le site γ.

Simon prouve (III.1.1) à l'aide d'une démonstration graphique, mais on peut également déduire (III.1.1) d'une classe d'inégalités plus générales établies par R. Boel et P. Kasteleyn [66,67].

L'inégalité initiale de Simon a été ensuite généralisée dans plusieurs directions. Aizenman et Simon l'ont étendu au modèle de Heisenberg à N composantes pour N = 2,3,4 [53], c'est à dire lorsqu'on peut disposer d'inégalités de type gaussien [70-72]. E. Lieb a obtenu dans le cas du modèle d'Ising une inégalité plus forte:

fonctions de corrélation lorsqu'il y a décroissance polynomiale suffisamment forte, on peut noter des bornes rigoureuses sur les températures critiques et les exposants critiques des modèles correspondants. Les conséquences des inégalités de Lieb sont plus intéréssantes encore. En particulier elles fournissent le seul algorithme connu qui permette en principe le calcul à n'importe quelle approximation de la température critique des modèles à une ou deux composantes. En pratique, toutefois, la convergence de l'algorithme est très lente et une précision de 10^{-2} ou 10^{-3} est sans doute hors d'atteinte, par ce moyen, des ordinateurs les plus puissants [52]!

Le principe de l'algorithme est cependant très simple. Il suffit de calculer une suite de températures T_n telles que

$$\sum_{\alpha \varepsilon \partial A_n} <\sigma_0 \sigma_\alpha>_{A_n} (T_n) = 1$$

pour un système formé d'une boîte finie A_n centrée à l'origine, de bord ∂A_n et de taille croissante avec n. La suite ainsi obtenue converge vers la température critique exacte du modèle [52].

Parmi les autres conséquences de l'inégalité de Lieb on peut mentionner des résultats concernant la continuité de la masse en fonction des paramètres de la théorie, en particulier une nouvelle démonstration du résultat de McBryan et Rosen [68], selon lequel la masse tend vers 0 lorsque la température décroît

Un peu plus tard Brydges, Fröhlich et Spencer, utilisant la représentation en chemins aléatoires dont il sera question dans la partie IV de cette thèse, ont tout à fait clarifié le rapport entre les inégalités gaussiennes de Newman et l'inégalité (III.1.2); ils ont pu obtenir ces inégalités pour des modèles à une ou deux composantes plus généraux que celui d'Ising ou des rotateurs plans, comprenant en particulier la théorie ϕ^4 à 1 ou 2 composantes sur réseau. La forme un peu différente de l'inégalité (III.1.2) obtenue par eux est la suivante, ressemblant à l'inégalité (4.1) de la référence [53]:

Si A est un ensemble de sites contenant α et F un polynôme dans la composante i des spins n'appartenant pas à A, on a

$$<\sigma_\alpha^i F> \leq \sum_{\substack{\beta\in A \\ \gamma\notin A}} <\sigma_\alpha^i \sigma_\beta^i>_A J_{\beta\gamma} <\sigma_\gamma^i F> \qquad \text{(III.1.4)}$$

où $J_{\beta\gamma}$ est le couplage entre les sites β et γ. Dans le cas d'une interaction constante entre plus proches voisins on retrouve une forme ressemblant à (III.1.2). En bonne logique il conviendrait donc sans doute de désigner ces inégalités sous le nom d'inégalités de Simon-Lieb-Aizenman -Rivasseau-Brydges-Fröhlich-Spencer! Nous allons rappeler rapidement les applications de ces inégalités en nous contentant de nommer inégalités de Simon celle du type (III.1.1) et de Lieb celles du type (III.1.2). Parmi les conséquences des inégalités de Simon, outre la décroissance exponentielle à l'infini des

jusqu'à la température critique.

Enfin l'une des conséquences les plus spectaculaires est sans doute la nouvelle preuve de l'existence d'un <<mass gap>> pour les théories ϕ_2^4 à faible couplage [69]. Tout en répondant au problème proposé par Simon [52], les auteurs de [55] n'ont plus besoin de la lourde machinerie de la <<cluster expansion à la Glimm, Jaffe, Spencer>>. Bien que limitée au cas de $\lambda\phi_2^4$ et non de $\lambda P(\phi)_2$, leur démonstration très rapide se contente d'utiliser les inégalités de Lieb pour obtenir la décroissance exponentielle à longue distance dans le cas du modèle faiblement couplé sur réseau; le résultat se transpose ensuite sans grande difficulté à la théorie continue dont on sait qu'elle est la limite de la théorie sur réseau lorsque le pas du réseau tend vers zéro.

En conclusion nous voudrions souligner qu'à notre avis les méthodes graphiques ou les développements en chemins aléatoires ne sont, dans ces démonstrations, que deux aspects voisins de la même idée, celle du développement à haute température, que nous allons aussi retrouver, encore un peu plus déguisée, dans la partie IV. Il n'est donc pas surprenant que les inégalités de corrélation obtenues par ces méthodes aient, jusqu'à présent uniquement conduit à des résultats concernant la région à haute température, à une seule phase et à décroissance exponentielle des corrélations, pour les modèles considérés.

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Lieb's Correlation Inequality for Plane Rotors

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Abstract. We prove a conjecture by E. Lieb, which leads to the Lieb inequality for plane rotors. As in the Ising model case, this inequality implies the existence of an algorithm to compute the transition temperature of this model.

Introduction

In [2], Simon proved and applied certain correlation inequalities. These inequalities are special cases of a class of inequalities proved earlier by Boel and Kasteleyn [3–6]. For a finite range pairwise interacting Ising ferromagnet on a lattice L, the inequalities in [2] are:

$$\langle s_a \cdot s_c \rangle \leqq \sum_{b \in B} \langle s_a \cdot s_b \rangle \langle s_b \cdot s_c \rangle , \tag{1}$$

where B is any set of spins separating a from c.

Lieb has generalized this inequality. In [1], he showed the stronger assertion:

$$\langle s_a \cdot s_c \rangle \leqq \sum_{b \in B} \langle s_a \cdot s_b \rangle_A \langle s_b \cdot s_c \rangle , \tag{2}$$

where A is the union of B and the connected component of $L-B$ containing a, and $\langle \cdot \rangle_A$ denotes expection values with respect to the A system only. He also reduced the proof of a similar inequality for plane rotors to a conjecture on directed graphs, which he proves in a special case, and which we prove generally in the next section. Hence we obtain:

Theorem. *Let us consider a plane rotors model with pairwise interaction between two spins* $\mathbf{s}_a$ *and* $\mathbf{s}_b$ *of type* $-J_{a,b}\mathbf{s}_a \cdot \mathbf{s}_b$, *with* $J_{a,b} \geqq 0$.
Then

$$\langle \mathbf{s}_a \cdot \mathbf{s}_c \rangle \leqq \sum_{b \in B} \langle \mathbf{s}_a \cdot \mathbf{s}_b \rangle_A \langle \mathbf{s}_b \cdot \mathbf{s}_c \rangle , \tag{3}$$

where B separates a from c, and A has same meaning as previously.

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Among the consequences of (3), is the existence of an algorithm to compute the transition temperature of this model (in the sense that above, but not below, there is exponential decay of the two point function), and the continuity of the mass gap as function of the interaction, for nearest neighbor interactions [1, 2].

Finally we remark that the method used to prove our lemma extends to prove other combinatorial results of the same kind. We intend to explore more completely in the future the consequences of these results for the plane rotors model.

I) We prove a slightly stronger result than the original conjecture of [1]:

Lemma. *Let G be a finite directed graph (possibly with several edges between two vertices) and let the valence at vertex M_i, i.e. the number of arrows in minus the number out of M_i, be m_i. Clearly, $\sum m_i = 0$. Suppose $m_1 > 0$, and $m_2, m_3, \ldots, m_k < 0, m_i \geqq 0$ otherwise. Let $N(G)$ be the set of subgraphs of G (subsets of edges), including the empty graph, having valence 0 at each vertex. Let $K(M_1, G)$ be the set of subgraphs of G with the following property: vertex M_1 has valence $+1$, some vertex M_i with $2 \leqq i \leqq k$ has valence -1, all other valences are 0.*
Then:

$$|N(G)| \leqq |K(M_1, G)| . \tag{4}$$

Proof. We prove the lemma by induction on the number of edges in G. The lemma is trivial for a graph with a unique edge. We suppose it is true up to l edges. Let G have $l+1$ edges, among which $l_1, \ldots, l_p$ are the arrows into M_1 and $l_{p+1}, \ldots, l_{p+q}$ are the arrows out of M_1.
We define:

$$\text{for} \quad 1 \leqq j \leqq p \qquad \begin{cases} N_j(G) = \{S \in N(G) : l_j \notin S\} \\ K_j(M_1, G) = \{S \in K(M_1, G) : l_j \in S\} \end{cases}$$

$$\text{for} \quad p+1 \leqq j \leqq p+q \begin{cases} N_j(G) = \{S \in N(G) : l_j \in S\} \\ K_j(M_1, G) = \{S \in K(M_1, G) : l_j \notin S\} . \end{cases}$$

We claim that for $1 \leqq j \leqq p+q$,

$$|N_j(G)| \leqq |K_j(M_1, G)| . \tag{5}$$

Assuming this, the lemma follows easily, since every subgraph in $N(G)$ belongs to exactly p subsets $N_j(G)$, and every subgraph in $K(M_1, G)$ belongs to exactly $q+1$ subsets $K_j(M_1, G)$. Hence:

$$p|N(G)| = \sum_j |N_j(G)| \leqq \sum_j |K_j(M_1, G)| = (q+1)|K(M_1, G)|$$

and $(q+1)/p \leqq 1$ is precisely the condition $m_1 > 0$.

To prove the claim, we consider first the case $1 \leqq j \leqq p$. Let l_j be the arrow $M_j M_1$. If $m_j < 0$, we have a natural injection of $N_j(G)$ into $K_j(M_1, G)$: to S we associate $S \cup \{l_j\}$. If $m_j \geqq 0$, we apply the induction hypothesis to the graph $G_j = G - \{l_j\}$: in G_j, $m_1 \geqq 0$, $m_j > 0$, hence $|N(G_j)| \leqq |K(M_j, G_j)|$. On the other hand, there are natural bijections of $N(G_j)$ into $N_j(G)$, and of $K(M_j, G_j)$ into $K_j(M_1, G)$ (respectively $S \to S$, and $S \to S \cup \{l_j\}$). In both cases, (5) is proved.

Finally, if $p+1 \leqq j \leqq p+q$, either $N_j(G)$ is empty, and (5) is trivial, or there exists a subgraph L_j containing l_j in $N_j(G)$. We then note the effect of reversing arrows in a given subgraph L belonging to $N(G)$. We call G^L the graph obtained from G by this transformation. The valences m_i do not change. Moreover, there are natural bijections between $N(G)$ and $N(G^L)$, and between $K(M_1, G)$ and $K(M_1, G^L)$, which to a subgraph S associate $S \Delta L$, where Δ is the symmetric difference of subsets. Returning to the proof of (5), we reverse the arrows in the subgraph L_j previously introduced, and transform the problem into one of the preceding kind for $G^L j$, hence already solved. Hence (5) holds in every case.

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III.3. Autres Résultats, Commentaires

Dans cette section, nous rappelons que l'idée de l'article qui précéde s'applique aussi à la démonstration d'autres résultats analogues de théorie des graphes (voir aussi notre preprint [56]). Nous résumons également une tentative, en collaboration avec F. Dunlop, infructueuse mais à notre avis instructive, en vue de généraliser les résultats de Lebowitz [58] au modèle du rotateur plan.

Notre point de départ est le résultat de théorie des graphes suivant:

Théorème III.3 ([56])

Soit G un graphe orienté avec deux vertex distingués A et B. Pour tout sous-graphe S de G (sous-ensemble de lignes de G, éventuellement vide) on note $v_S(M)$ le nombre de lignes de S entrant en M moins le nombre de lignes de S sortant de M.

Supposons que $v_G(A) \leq +1$

$$v_G(M) \leq 0 \quad \forall\, M \neq A. \tag{III.3.1}$$

Alors en posant

$$K = \{S \subset G:\ v_S(A) = +1,\ v_S(B) = -1,\ v_S(M) = 0\ \ \forall M \neq A,B\}$$

$$N = \{S \subset G;\ v_S(M) = 0\ \ \forall M\ \} \tag{III.3.2}$$

on a $|K| \leq |N|$ (III.3.3)

Preuve Bien que la démonstration soit calquée sur celle de l'article (III.2), il ne nous semble pas que l'on puisse déduire directement l'un des résultats de l'autre.

La preuve se fait par récurrence sur le nombre de lignes de G. Le résultat est trivial si G n'a qu'une ligne. Nous le supposons vrai jusqu'à ℓ lignes, et considérons un graphe G à $\ell+1$ lignes. Soient $\ell_1,\ldots,\ell_p$ les lignes arrivant en A et $\ell_{p+1},\ldots,\ell_q$ celles quittant A. On définit:

$$\begin{cases} \text{pour } 1 \le i \le p & \begin{cases} K^i = \{S \in K : \ell_i \in S\} \\ N^i = \{S \in N : \ell_i \notin S\} \end{cases} \\ \text{pour } p+1 \le i \le q & \begin{cases} K^i = \{S \in K : \ell_i \notin S\} \\ N^i = \{S \in N : \ell_i \in S\} \end{cases} \end{cases}$$

Montrons que: $|K^i| \le |N^i| \quad \forall i,\ 1 \le i \le p+q$ (III.3.4)

En effet considérons $\ell_i = \overrightarrow{A_i A}$ pour $1 \le i \le p$. Si $A_i = B$, l'application $S \to S - \{\ell_i\}$ est une bijection de K^i sur N^i Si $A_i \neq B$, on applique l'hypothèse de récurrence au graphe $G_i = G - \{\ell_i\}$ avec les vertex distingués A_i et B. Comme $v_G(A_i) \le 0$, $v_{G_i}(A_i) \le +1$, et toutes les autres valences ont diminué. Si on note K_i et N_i les ensembles définis par (III.3.2) en remplaçant G, A et B respectivement par G_i, A_i et B, l'hypothèse de récurrence fournit $|K_i| \le |N_i|$ et donc $|K^i| \le |N^i|$

puisqu'il existe des bijections naturelles entre K_i et K^i et entre N_i et N^i (respectivement $S \to S \cup \{\ell_i\}$ et $S \to S$).
Si $p+1 \leq i \leq p+q$, il existe un sous graphe $L_i \in N$, avec $\ell_i \in L_i$, sans quoi on pourrait montrer l'existence d'un vertex de G, autre que A, de valence strictement positive ce qui contredirait (III.3.1). On utilise alors l'argument qui précède, appliqué au graphe G^{L_i} obtenu en renversant les flèches de L_i (même technique que en (III.2)).

Comme tout élément de K appartient à exactement p ensembles N^i, on conclut:

$$(q+1) \quad |K| = \sum_i |K^i| \leq \sum_i |N^i| = p|N|$$

ce qui achève la preuve de (III.3.3) puisque $v_G(A) \leq 1 \Leftrightarrow \frac{p}{q+1} = 1$.

Avec François Dunlop, nous avons essayé d'appliquer le théorème III.3 à l'étude du rotateur plan à basse température. On s'attend à ce que la structure des pures phases (états de Gibbs extrêmaux invariants par translation) soit un simple cercle, c'est à dire que ces phases soient indexées par un angle $\theta \in [0,2\pi]$ qui n'est autre que la magnétisation spontanée , lorsque la dimension du réseau est 3 ou plus [60]. Pour obtenir ce résultat il suffirait de prouver que la valeur moyenne des observables invariantes par une rotation globale de tous les spins du réseau est indépendante de la pure phase considérée. On en déduit alors le résultat cherché en faisant appel à la structure

de simplexe de Choquet des états de Gibbs [73-75], par un argument parallèle à celui de Lebowitz pour le modèle d'Ising [58]. On s'attend dans le cas du rotateur plan à ce qu'il n'y ait pas de transition de phase de première espèce, donc à ce que l'énergie par site soit continue dans la température. D'après un argument de convexité on sait que ceci est au moins vrai partout sauf peut-être pour un ensemble dénombrable B de valeurs de $\beta = (kT)^{-1}$. Pour $\beta \notin B$, on obtient alors l'indépendance de la valeur moyenne de $\cos(\theta_i - \theta_j)$ où θ_i et θ_j sont les variables angulaires associées aux deux sites i et j, par le même argument que dans [58]. On peut donc essayer de suivre la même idée que Lebowitz c'est à dire de trouver une inégalité de corrélation permettant, par récurrence sur le nombre des facteurs, de démontrer l'indépendance d'un produit quelconque de facteurs $\cos(\theta_i - \theta_j)$, puis par un argument de densité d'en déduire l'indépendance de toute observable invariante par rotation globale des spins. Par "indépendance" nous entendons indépendance de la valeur moyenne de la quantité considérée par rapport à la pure phase dans laquelle on l'évalue.

Une inégalité de corrélation qui suffirait pour cela serait la généralisation suivante de l'inégalité de Lebowitz, différente de celle obtenue dans la référence [60]:

Conjecture III.3.1

Considérons deux copies du même modèle de rotateur plans sur un réseau fini Λ, avec les mesures de Gibbs correspondantes:

$$\mu(d\theta) = Z^{-1} \exp \sum_{<i,j>} J_{ij} \cos(\theta_i - \theta_j)$$

$$\mu'(d\theta) = Z^{-1} \exp \sum_{<i,j>} J'_{ij} \cos(\theta_i - \theta_j)$$

Il existe une constante $c > 0$ telle que, si $J_{ij} \geq |J'_{ij}|$ pour toute paire i,j. on ait:

$$\int d\mu \prod_{<ij>} [\cos(\theta_i - \theta_j)]^{m_{ij}} - \int d\mu' \prod_{<ij>} [\cos(\theta_i - \theta_j)]^{m_{ij}} \geq$$

$$c \Big| \Big\{ \int d\mu \cos(\theta_k - \theta_\ell) \prod_{<ij>} [\cos(\theta_i - \theta_j)]^{m_{ij}} \Big\} \Big\{ \int d\mu' \cos(\theta_k - \theta_\ell) \Big\} -$$

$$\Big\{ \int d\mu \cos(\theta_k - \theta_\ell) \Big\} \Big\{ \int d\mu' \cos(\theta_k - \theta_\ell) \prod_{<ij>} [\cos(\theta_i - \theta_j)] \Big\}^{m_{ij}} \Big| \qquad \text{(III.3.5)}$$

pour tout choix d'entiers $m_{ij} \geq 0$ et des sites k et ℓ.

Si l'on développe en puissances de l'interaction, cette conjecture serait une conséquence de l'inégalité suivante, malheureusement fausse:

Conjecture fausse III.3.2

Il existe une constante $c > 0$ telle que pour tout choix d'entiers $m_{ij} \geq 0$, $n_{ij} \geq 0$ et $\varepsilon = \pm 1$, $1 \leq k < \ell \leq n$:

$$\int - \int_0^{2\pi} d\theta_1 .. \, d\theta_n \, d\theta'_1 .. d\theta'_n \prod_{1\le i<j\le n} [\cos(\theta_i-\theta_j) + \cos(\theta'_i-\theta'_j)]^{m_{ij}}$$

$$[\cos(\theta_i-\theta_j) - \cos(\theta'_i-\theta'_j)]^{n_{ij}} \cdot$$

$$\cdot [1 + c\varepsilon \cos(\theta_k-\theta_\ell) \cos(\theta'_k-\theta'_\ell)] \ge 0 \qquad \text{(III.3.6)}$$

Remarquons que la conjecture III.3.2, si elle était vraie, serait une version renforcée des inégalités de Ginibre [57], qui correspondent dans ce cas à (III.3.6) avec c = 0.
En décomposant, dans l'inégalité (III.3.6) chaque cosinus en somme d'exponentielles imaginaires on obtient l'inégalité équivalente suivante, en termes de graphes orientés:

Conjecture fausse III.3.3 (forme graphique de la conjecture III.3.2)

Il existe une constante c > 0, telle que si G est un graphe orienté avec les propriétés suivantes:

a) $v_G(M) = 0 \quad \forall$ M vertex de G

b) A toute ligne de G est associé un signe + ou -

c) G a deux vertex distinqués A et B

alors on a :

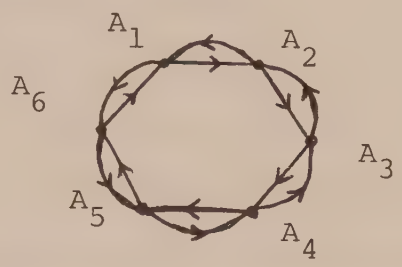
$$c|K^+-K^-| \cdot |N^+-N^-| \le |N^+-N^-|^2 \qquad \text{(III.3.7)}$$

où K^+ et N^+ (respectivement : K^- et N^-) sont les sous ensembles de K et de N contenant un nombre pair (resp: impair) de lignes portant le signe +. (Pour la définition de $v_G(M)$, K et N, voir le théorème III.3.).

Nous avons trouvé une famille de graphes constituant un contre-exemple à la conjecture (III.3.3) (et donc III.3.2).

Considérons le graphe Gp, $p \geq 3$ comportant 2p lignes et p vertex qui est formé d'une chaîne de p boucles élémentaires. Par exemple:

G_6 =



Les lignes et les vertex sont numérotés de sorte que la ligne i va de A_i à A_{i+1} pour $1 \leq i \leq p$ et de A_{i-p+1} à A_{i-p} pour $p + 1 \leq i \leq 2p$. La bulle élémentaire formée des lignes i et p + i pour $1 \leq i \leq p$ sera notée b_i; l'ensemble des telles bulles est noté E. Alors l'ensemble N_p correspondant au graphe G_p est formé de toutes les parties de E et des deux ensembles exceptionnels e_1 et e_2 formés, respectivement, des lignes i pour $1 \leq i \leq p$ et des lignes i pour $p+1 \leq i \leq 2p$. Si l'on considère A_3 et A_1 comme vertex distingués A et B dans l'énoncé de la conjecture (III.3.3) et si on associe un signe + à toutes les lignes i sauf à i = 1 et i = 2, on obtient aisément:

$$N_p^+ - N_p^- = 2 \quad \forall p. \quad \text{car l'application } P \to P \Delta \{b_1\}$$

est une bijection entre les sous ensembles de N_p^+ et N_p^- formés de parties de E, et on remarque que e_1 et e_2 appartiennent

à N_p^+. D'autre part K se décompose en:

- les sous ensembles contenant les lignes 1 et 2 et une partie quelconque de $E - \{b_1, b_2\}$
- les sous ensembles contenant les lignes i pour $p+3 \leq i \leq 2p$, et une partie quelconque de $\{b_1, b_2\}$

Il est donc aisé de vérifier que:

$$K_p^+ - K_p^- = 2^{p-2}$$

ce qui achève de ruiner la conjecture (III.3.3).

On s'aperçoit toutefois facilement que l'inégalité (III.3.7) est vraie dans un certain nombre de cas:

a) lorsque $N^+ = N^-$, c'est à dire lorsque l'inégalité de Ginibre est saturée.

b) Lorsque $N^+ = 0$ ou $N^- = 0$. Dans ces deux cas (III.3.7) découle du théorème III.3. En particulier la conjecture III.3.2 est vraie si $m_{ij} = 0 \ \forall <ij>$ ou si $n_{ij} = 0 \ \forall <i,j>$.

c) Si l'on permet $c = 0$ ou mieux, $c = 2^{-\ell(G)}$ variant avec le nombre de lignes de G on obtient des formes affaiblies et vraies de la conjecture; dans le premier cas on a une démonstration graphique (nouvelle, à notre connaissance) des inégalités de Ginibre : dans le deuxième cas on obtient un renforcement à notre avis intéréssant de ces inégalités mais que nous n'avons pas encore su utiliser.

Des résultats partiels permettent d'espérer que la conjecture (III.3.1) est vraie, bien que l'inégalité plus forte lui correspondant graphe par graphe soit fausse. Il s'agirait alors d'une

situation analogue à celle des inégalités de Griffiths pour le modèle de Heisenberg à plus de deux composantes: on espère que ces inégalités sont vraies bien qu'on sache que les inégalités plus fortes de Ginibre qui leur correspondent sont violées pour certains graphes [61].

Nous conclurons en rappelant donc que la structure des pures phases du modèle du rotateur plan à basse température (en champ extérieur nul) reste un intéréssant problème non élucidé; il serait important aussi de pouvoir prouver dans ce cas l'absence très probable de transition de phase de première espèce.

Note: Peu de temps avant le tirage de cette thèse, Charles Pfister a démontré une inégalité analogue à III.3.1, et résolu complètement le problème proposé dans ce paragraphe. Il a ingénieusement remplacé $1 + c\varepsilon \cos(\theta_k-\theta_\ell) \cos(\theta'-\theta'_\ell)$ par $\exp c\varepsilon \cos(\theta_k-\theta_\ell) \cos(\theta'_k-\theta'_\ell)$ dans l'inégalité III.3.6. La conjecture III.3.2 ainsi modifiée devient alors correcte. Prenant dans un second temps la limite $\varepsilon \to 0$, il obtient essentiellement notre conjecture III.3.1, confirmant l'idée exprimée plus haut. L'analyse de Pfister élucide de plus complètement, pour les valeurs régulières de β, la structure des états de Gibbs invariants par translation dans le cas de systèmes à groupe de symétrie abélien plus généraux, en particulier lorsque la symétrie n'est que partiellement brisée.

Référence: C. E. Pfister, preprint, avril 82.

QUATRIEME PARTIE

Développement en 1/n et Représentation en chemins aléatoires

"La vérité
fera de vous
des hommes libres."

Jean, 8,32

BOREL SUMMABILITY OF THE 1/N EXPANSION FOR THE N-VECTOR [O(N) NON-LINEAR σ] MODELS

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ABSTRACT

We construct an analytic interpolation in 1/N for the N-vector [O(N) non-linear σ] models with N-component fields o a lattice. This interpolation, valid at sufficiently high temperatures, extends over a large domain in the complex plan containing the half plane Re(1/N) > 0. We use this result to show that the 1/N expansion of the free energy density and of the correlation functions is Borel summable in the thermodynamic limit and at high temperature.

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1. Introduction, Notations and Main Results

In this paper we continue a mathematically rigorous analysis of the 1/N expansion in the N-vector models, initiated by A. Kupiainen [1,2]. Kupiainen has shown that the 1/N expansion is asymptotic for two families of models, the N-vector models on a simple, (hyper) cubic lattice $\mathbb{Z}^d$, d = 2,3,4,..., at temperatures above the critical temperature of the spherical model (N = ∞), and a class of weakly coupled N-component $\lambda|\vec{\phi}|^4$ models in two space-time dimensions. A careful analysis of the 1/N expansion for the three-dimensional O(N) σ-models in the continuum limit has been carried out by I. Aref'eva [3] who, however, has not determined its nature. For a summary of the history of 1/N expansions and references to important, earlier work, see Kupiainen's papers [1,2].

A natural problem is to study the analyticity properties in 1/N and to determine the summability properties of the 1/N expansion for the models mentioned above. Billionnet and Renouard have recently proven that the 1/N expansion for weakly coupled N-component $\lambda|\vec{\phi}|^4$ models in two dimensions is Borel-summable [4]. In this paper we establish the same result for the O(N) non-linear σ-models on a lattice of arbitrary dimension, at high temperature. The methods used in this paper are different from the ones in [4]. In [4] the main technical difficulty appears in the construction of the continuum (ultraviolet)

limit. Here we do not construct the continuum limit. However, the 1/N expansion for the N-vector models is an expansion around a stationary point of a complex measure which has only "poor fall-off at infinity" and which is hard to control when 1/N is close to the imaginary axis. We resolve this difficulty by superimposing a second expansion related to a standard high temperature expansion. Each term in this double expansion can be calculated explicitly and turns out to be analytic in 1/N everywhere except on the interval [-1/2,0]. Our double expansion can be interpreted as a random walk -- or polymer representation of thermodynamic and correlation functions of the N-vector models. It is inspired by work of Symanzik [5] and a rigorous version thereof [6] which has recently been applied intensively to the study of the Ising (N=1) and classical rotor (N=2) models [6-8]. The representation described in [6] can be interpreted as a double expansion in powers of N and β. It is used in [9] to derive joint analyticity in N and β near $N = 0$, $\beta = 0$. As in [6], our double expansion represents the N-vector models as gases of random walks or polymer chains with soft core interaction. In contrast to the random walks used in [6], the random walks introduced in this paper make steps of arbitrary length, but with exponentially decaying probability. Our random walk representation is chosen in such a way that the spherical model limit, $N \to \infty$, is reached smoothly along rays in the half plane $\mathrm{Re}\, 1/N > 0$.

The convergence of our double expansion is controlled by cluster (or polymer) expansion methods described in [10]; see also [11,12].

We establish convergence at high temperature in a large domain of the 1/N plane, uniformly in the volume cutoff. We also derive bounds on the k^{th} derivative in 1/N of thermodynamic and correlation functions. When combined with a theorem due to Nevanlinna and Sokal [13] they suffice to prove Borel summability of the 1/N expansion.

Our paper is organized as follows:

In the remainder of Sect. 1 we define the lattice N-vector models, introduce some notation and summarize our main results in the form of a theorem.

In Sect. 2 we derive the basic random walk representation of our models and express it in a compact form which makes the methods of [10,12] accessible.

In Sect. 3 we recall the cluster expansion [10-12] and use it to control the convergence of our random walk representation. We construct the thermodynamic limit of our models and verify analyticity in 1/N in a large domain and prove Borel summability at $1/N = 0$, for sufficiently high temperatures; (see Theorem A below).

In Sect. 4 we discuss our results and describe some open problems related to the main theme of this paper.

Some technical estimates needed in Sects. 2, 3 are proven in an appendix.

Throughout this paper we follow quite closely the notation introduced in [1]. We adopt the conventions that

$$\text{an empty sum} = 0 ,$$

$$\text{an empty product} = 1 .$$

With each site, j, of the simple (hyper) cubic lattice $\mathbb{Z}^d$, $d = 2,3,4,\ldots$, we associate a classical, N-component spin

$$\phi_j = (\phi_j^1,\ldots,\phi_j^N) , \quad N = 1,2,3,\ldots \tag{1.1}$$

of length N, i.e. $|\phi_j|^2 = N$. The Hamilton function of the model in a finite sublattice Λ of $\mathbb{Z}^d$ is given by

$$H_\Lambda(\phi) = - \sum_{(jj')\subset\Lambda} \phi_j \cdot \phi_{j'} , \tag{1.2}$$

where the summation ranges over all pairs of nearest neighbors in Λ, and $\phi \cdot \phi'$ is the usual scalar product of ϕ with ϕ'. For reasons of technical simplicity we shall choose Λ to be a rectangle and impose periodic boundary conditions, but (in

contrast to the methods used in [1]) we can accommodate arbitrary boundary conditions. The Gibbs state of the N-vector model (= Euclidean vacuum functional of the O(N) non-linear σ-model on the lattice) on the sublattice Λ is given by

$$d\mu_\beta^{(\Lambda)}(\phi) = \left[Z_\beta^{(\Lambda)}\right]^{-1} e^{-\beta H_\Lambda(\phi)} \prod_{j\in\Lambda} \delta(|\phi_j|^2 - N) d^N\phi_j \,, \qquad (1.3)$$

where β is the inverse temperature (= inverse square coupling constant), and $Z_\beta^{(\Lambda)}$ is the usual partition function.

The correlation functions, $\left\langle \phi_{x_1}^{\alpha_1} \ldots \phi_{x_m}^{\alpha_m} \right\rangle_{\beta,z}$, $z = 1/\sqrt{N}$, are obtained from the characteristic functional

$$S(g) = \int d\mu_\beta^{(\Lambda)}(\phi) \exp \phi(g) \,, \qquad (1.4)$$

$$\phi(g) \equiv \sum_{\alpha=1}^{d} \sum_j \phi_j^\alpha g_j^\alpha$$

by functional differentiation in g.

Following [7,1,6] we now introduce the dual representation of the N-vector model which displays N, or 1/N , as a parameter. Let Δ be the usual finite difference Laplacian defined by

$$\Delta_{ij} = -2d\,\delta_{ij} + \sum_{|k|=1} \delta_{i,j+k} \quad ,$$

with some boundary conditions (e.g. periodic) imposed at $\partial\Lambda$. We define

$$C \equiv C_\beta = (-\beta\Delta + m^2)^{-1} \quad , \tag{1.5}$$

where $m \equiv m_j \geq 0$ is some mass parameter which may, a priori, depend on the site $j \in \Lambda$ and will be specified below. We set

$$z = 1/\sqrt{N} \quad \text{and} \quad y = 2/N = 2z^2 \tag{1.6}$$

The first step in the derivation of the dual representation is to notice that in the definition of $d\mu_\beta^{(\Lambda)}$, Eq. (1.3), we may multiply each factor $\delta(|\phi_j|^2 - N)$ by $\exp[-(\beta d + m_j^2/2)|\phi_j|^2]$ which can be absorbed in a redefinition of $Z_\beta^{(\Lambda)}$. Second, we Fourier-decompose $\delta(|\phi_j|^2 - N)$,

$$\delta(|\phi_j|^2 - N) = (1/2\pi)\int d\alpha_j e^{i\alpha_j(|\phi_j|^2 - N)}$$

for each $j \in \Lambda$, and subsequently interchange integration over α and ϕ. The resulting ϕ-integral is Gaussian, and we obtain

$$S(g) = Z_1^{-1} \int \exp\{(1/2)(g,[-\beta\Delta + m^2 - 2i\alpha]^{-1} g)\} \cdot \cdot \det(-\beta\Delta + m^2 - 2i\alpha)^{-N/2} \prod_{j\in\Lambda} e^{-i\alpha_j N} d\alpha_j \tag{1.7}$$

where $Z_1 = \text{const.}^{|\Lambda|} Z_\beta^{(\Lambda)}$, and $|\Lambda|$ is the number of sites in Λ. Identity (1.7) is very convenient for analyzing the behaviour of $S(g)$ near $N = 0$; (one can prove joint analyticity in β and N inside some discs centered at $N = 0$ and $\beta = 0$. See [9]). In order to determine the behaviour of $S(g)$ near $N = \infty$, one changes variables,

$$\alpha_j = z a_j, \text{ for all } j \in \Lambda,$$

and applies the identity

$$\det(A) = \exp \operatorname{tr} \log(A)$$

This yields

$$S(g) = \int \exp\{(1/2)(g,[1-2izC_\beta a]^{-1}C_\beta g)\}d\mu_{\beta,z}(a) , \qquad (1.8)$$

where

$$d\mu_{\beta,z}(a) = Z_2^{-1} \exp\{-(1/y)\ \mathrm{tr}\ \log(1 - 2izC_\beta a)\} \cdot \cdot \prod_{j\in\Lambda} \exp(-ia_j/z)da_j , \qquad (1.9)$$

and

$$Z_2 = \mathrm{const.}^{|\Lambda|} Z_1 .$$

We now choose $m^2 = m_j^2$ such that the terms linear in a in the exponents on the r.s. of (1.9) cancel. This is the case iff

$$C_{\beta,jj} = 1, \quad \text{for all} \quad j \in \Lambda , \qquad (1.10)$$

where $C_{\beta,ij} = (\delta_i, C_\beta \delta_j)$.

This condition ensures that the complex measure $d\mu_{\beta,z}$ is , in zeroth order in z , given by the positive Gaussian measure of

the spherical model, the deviation from the Gaussian being of the form exp O(z). Condition (1.10) plays a crucial rôle in our subsequent analysis. It is a system of transcendental equations for the mass parameters $m_j = m_j(\beta,\Lambda)$. The simplifying feature of periodic b.c. is that condition (1.10) reduces to a single equation (because of translation invariance) which has a constant solution $m_j = m(\beta,\Lambda) > 0$, for all β, when $|\Lambda|$ is finite. This is the only instance in this paper where periodic b.c. are more convenient than other b.c. In [1], $m = m(\beta)$ is chosen so that (1.10) is only satisfied in the thermodynamic limit, $\Lambda \to \mathbb{Z}^d$. It then reduces to

$$\int_{[-\pi,\pi]^d} \frac{d^d p}{(2\pi)^d} \left[2\beta \sum_{\mu=1}^{d} (1 - \cos p_\mu) + m^2 \right]^{-1} = 1 \qquad (1.11)$$

The inverse critical temperature β_S of the spherical model is the maximal value of β for which (1.11) has a solution, $m = m(\beta_S) = 0$. For $d \geq 3$, $0 < \beta_S < \infty$, while for $d = 2$, $\beta_S = \infty$. For $\beta < \beta_S$, (1.11) has a solution $m = m(\beta)$, and the solution, $m(\beta,\Lambda)$, of (1.10) (for periodic b.c.) converges to $m(\beta)$, as $\Lambda \to \mathbb{Z}^d$, for all $\beta < \beta_S$. Moreover

$$\lim_{\beta \to 0} m(\beta,\Lambda) = \sup_{\beta} m(\beta,\Lambda) = 1 .$$

Note that on the r.s. of (1.8) and (1.9) $z = 1/\sqrt{N}$ appears as a parameter. It is not hard to see that $S(g)$ is a function of $y = 2z^2$ and that it is, in fact, analytic in y in a complex neighborhood of the open positive half axis when $|\Lambda|$ is finite. Moreover when $y = 2/N$ and N is an integer it agrees with the functional defined in (1.4). Our task is now to continue $S(g) = S^{(\Lambda)}_{\beta,y}(g)$ analytically in y (and β) to as large a domain as possible and to derive estimates on $|S^{(\Lambda)}_{\beta,y}(g)|$ in that domain which are uniform in Λ. Our main results can be summarized as follows.

Theorem A.

For $0 < \beta < \beta_\varepsilon$, for some positive constant $\beta_\varepsilon < \beta_S$, the thermodynamic limits of the free energy and the correlation functions exist and are analytic functions of $y = 2/N$ on the domain $\{y : |\arg y| < \pi-\varepsilon$, or $|y| > 1+\varepsilon\}$. In that range of parameters connected correlation functions have exponential clustering. For $\beta < \beta_{\frac{\pi}{2}}$, the $1/N$ expansions of the free energy and the correlation functions are Borel summable.

2. A New Random Walk Representation and the Polymer Method

We combine Eqs. (1.8) and (1.9) into the following formula for $S(g)$:

$$S(g) = Z_2^{-1} \int \exp\{(1/2)(g,[1-2izC_\beta a]^{-1}C_\beta g)\} \cdot \exp\{-(1/y)\mathrm{tr}\ \log(1-2izC_\beta a)\} \prod_{j\in\Lambda} \exp(-ia_j/z)da_j \,, \tag{2.1}$$

where Z_2 is chosen such that

$$S(g=0) = 1 \,. \tag{2.2}$$

The basic idea of this section is to split C_β into a diagonal part C_β^D and an off-diagonal part C_β^0 and to expand perturbatively in C_β^0. By (1.10)

$$C^D_{\beta,ij} = C_{\beta,jj}\,\delta_{ij} = \delta_{ij} \tag{2.3}$$

In order to understand the crucial significance of condition (2.3) in our scheme we first consider the case where

$$C_{\beta,jj} = \gamma \,. \tag{2.4}$$

for some parameter $\gamma > 0$ which will subsequently be set = 1 .

We now insert the identity

$$1 - 2izC_\beta a = 1 - 2iz\gamma a - 2izC_\beta^0 a$$
$$= (1-2iz\gamma a)\left[1-\gamma^{-1}C_\beta^0\left(\frac{1}{1-2iz\gamma a} - 1\right)\right] \tag{2.5}$$

into (2.1) and expand in powers of $\gamma^{-1}C_\beta^0$. We make use of the identities

$$\log(1-2izC_\beta a) = \log(1-2iz\gamma a)$$
$$\left\{ - \sum_{k=1}^{\infty} \frac{1}{k}\left[\gamma^{-1}C_\beta^0\left(\frac{1}{1-2iz\gamma a} - 1\right)\right]^k \right\} , \tag{2.6}$$

and

$$[1-2izC_\beta a]^{-1}C_\beta = (1-2iz\gamma a)^{-1}\left[1-\gamma^{-1}C_\beta^0\left(\frac{1}{1-2iz\gamma a} - 1\right)\right]^{-1} C_\beta$$

$$= \gamma(1-2iz\gamma a)^{-1}\left\{ 1 + \sum_{k=0}^{\infty}\left[\gamma^{-1}C_\beta^0\left(\frac{1}{1-2iz\gamma a} - 1\right)\right]^k \cdot \right.$$

$$\left. \cdot\ \gamma^{-1}C_\beta^0\left(1-2iz\gamma a\right)^{-1} \right\} \tag{2.7}$$

Before we present the result of these expansions we introduce some notations and conventions:

An oriented random loop, denoted by ω, is a closed, directed random walk built of an arbitrary number, $p = p(\omega)$, of steps of arbitrary, but strictly positive length. Thus, an oriented random loop is a class of ordered sequences of jumps, from a site i_1 to a site i_2, from i_2 to i_3, and so on, the last jump being from i_p to $i_{p+1} = i_1$, but two sequences of jumps which only differ in the choice of the starting site correspond to the same random loop.

An open random walk, ω, from $x \in \Lambda$ to $u \in \Lambda$ consists of a sequence of $p = p(\omega)$ jumps, from $i_1 = x$ to i_2, from i_2 to i_3, and so on, and finally from i_p to $i_{p+1} = u$; $i_k \neq i_{k+1}$, for all k. If $x \neq u$ we have $p(\omega) \geq 1$. The circumstance that ω starts at x and ends at u is expressed, symbolically, as $\omega : x \to u$.

With every random walk, closed or open, we associate a weight, J_ω,

$$J_\omega = \prod_{k=1}^{p(\omega)} \left[\gamma^{-1} C_{\beta, i_k i_{k+1}} \right] \tag{2.8}$$

Furthermore, we let $n_j(\omega)$ denote the number of visits of a random walk ω at site j. If $\omega : x \to u$ is open, we define $n_j(x)$

and $n_j(u)$ to be the total number of visits of ω at x, u, respectively, not counting the first visit of ω at x, the last visit of ω at u, respectively.

The length, $\ell(\omega)$, of a random walk ω is defined by

$$\ell(\omega) = \sum_{k=1}^{p(\omega)} |i_k - i_{k+1}| , \tag{2.9}$$

where $|i-j|$ is the minimal number of nearest neighbor jumps necessary to get from i to j. Clearly

$$\ell(\omega) \geq p(\omega) \tag{2.10}$$

Inserting (2.5), (2.6) into the equation defining Z_2 (see (2.2)) and making use of definition (2.8) we obtain

$$Z_2 = \sum_{m=0}^{\infty} \sum_{\omega_1,\ldots,\omega_m} \frac{1}{y^m m!} \prod_{k=1}^{m} J_{\omega_k} \cdot$$

$$\cdot \int \prod_{j\in\Lambda} (1-2iz\gamma a_j)^{-1/y} \left(\frac{1}{1-2iz\gamma a_j} - 1 \right)^{n_j} e^{-ia_j/z} da_j , \tag{2.11}$$

where $\omega_1,\ldots,\omega_m$ are m not necessarily distinct oriented random loops and

$$n_j = \sum_{k=1}^{m} n_j(\omega_k) \ . \tag{2.12}$$

If we insert identities (2.6) and (2.7) into the r.s. of Eq. (2.1) defining S(g), differentiate twice with respect to g and make use of (2.8) we obtain the following expression for the two-point correlation, or Schwinger function

$$\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z} = z_2^{-1} \sum_{m=0}^{\infty} \sum_{\omega,\omega_1,\ldots,\omega_m} \frac{\gamma}{\gamma^m m!} J_\omega \prod_{k=1}^{m} J_{\omega_k} \cdot$$

$$\cdot \int \prod_{j\in\Lambda} \left(1-2iz\gamma a_j\right)^{-(1/\gamma)-\delta_{xj}-\delta_{uj}} \left(\frac{1}{1-2iz\gamma a_j} - 1 \right)^{n_j} e^{-ia_j/z} da_j \ , \tag{2.13}$$

where $\sum_{\omega,\omega_1,\ldots,\omega_m}$ ranges over all open random walks $\omega : x \to u$ and over m not necessarily distinct, oriented random loops $\omega_1,\ldots,\omega_m$, and

$$n_j = n_j(\omega) + \sum_{k=1}^{m} n_j(\omega_k) \tag{2.14}$$

If in (2.13) $x = u$, ω may be an empty random walk, in which case the exponent $-(1/\gamma)-\delta_{xj}-\delta_{uj}$ on the r.s. of (2.13) must be replaced by $-(1/\gamma)-\delta_{xj}$. Formulas analogous to (2.13) can be derived for an arbitrary 2n point correlation function, $\left\langle \phi_{x_1}^{\alpha_1} \ldots \phi_{x_{2n}}^{\alpha_{2n}} \right\rangle_{\beta,z}$, (with even numbers of α_j's equal to some $\alpha = 1,\ldots,N$). See also [6] for derivations of formulas quite similar to (2.11) and (2.13). For reasons of simplicity of our exposition we only study the two-point function, $\left\langle \phi_x^1 \phi_u^1 \right\rangle_{\beta,z}$, with $x \neq u$, henceforth, but our methods cover general correlations, as well. Now, note that the integrals over the parameters a_j, $j \in \Lambda$, in (2.11) and (2.13) factorize. They can be evaluated by using the simple identity

$$\int_{-\infty}^{\infty} da\, e^{-ia/z}(1-2iz\gamma a)^{-r}$$

$$= \frac{1}{\Gamma(r)} \int_0^{\infty} dt\, t^{r-1} e^{-t} \int_{-\infty}^{\infty} e^{ia(2z\gamma t - z^{-1})} da$$

$$= \frac{2\pi z}{\Gamma(r)} (2z^2\gamma)^{-r} \exp[-1/2z^2\gamma] . \qquad (2.15)$$

Let $\Omega = \{\omega_1,\ldots,\omega_m\}$ be an arbitrary, ordered m-tuple of oriented random loops, $|\Omega| \equiv m$, $m = 1,2,3,\ldots$, and

$$J_\Omega = \prod_{k=1}^{m} J_{\omega_k} \cdot \tag{2.16}$$

We now insert the r.s. of (2.15) and (2.16) into the r.s. of (2.11) and of (2.13) and use repeatedly the identity

$$\Gamma(r) = (r-1)\Gamma(r-1) .$$

This yields

$$Z_2 = \sum_{\Omega} \frac{1}{y^{|\Omega|}|\Omega|!} J_\Omega \prod_{j\in\Lambda} \left[\left(\Gamma(1/y)^{-1} 2\pi z e^{-1/y} (1/y)^{1/y} \right) \cdot \right.$$

$$\left. \cdot \left(\sum_{\ell=0}^{n_j} \binom{n_j}{\ell} (1/\gamma)^{\ell} (-1)^{n_j-\ell} \prod_{k=0}^{\ell-1} (1+ky)^{-1} \right) \right] , \tag{2.17}$$

and

$$Z_2 \left\langle \phi_x^1 \phi_u^1 \right\rangle_{\beta,z} = \sum_{\substack{\Omega \\ \omega: x \to u}} \frac{\gamma}{y^{|\Omega|}|\Omega|!} J_\Omega J_\omega \cdot$$

$$\cdot \prod_{j\in\Lambda} \left[\left(\Gamma(1/y)^{-1} 2\pi z e^{-1/y} (1/y)^{1/y} \right) \cdot \right.$$

$$\left. \cdot \sum_{\ell=0}^{n_j} \binom{n_j}{\ell} (1/\gamma)^{\ell+\delta_{xj}+\delta_{uj}} (-1)^{n_j-\ell} \prod_{k=\delta_{xj}+\delta_{uj}}^{\ell-1+\delta_{xj}+\delta_{uj}} (1+ky)^{-1} \right] \tag{2.18}$$

The factor $\left[\Gamma(1/y)^{-1}2\pi z e^{-1/y}(1/y)^{1/y}\right]^{|\Lambda|}$ can be absorbed in a redefinition of the partition function

$$Z_3 \equiv Z_2\left[\Gamma(1/y)^{-1}2\pi z e^{-1/y}(1/y)^{1/y}\right]^{-|\Lambda|} \tag{2.19}$$

We notice that that factor is the total volume of the spheres over which the classical spins of the system may range, normalized so that the limit, as $y \to 0$, is finite.

If we define free energy densities

$$\left.\begin{aligned} \beta f'_\Lambda(\beta,z) &= \frac{1}{|\Lambda|}\log Z_2\,, \\ \beta f_\Lambda(\beta,y) &= \frac{1}{|\Lambda|}\log Z_3 \end{aligned}\right\} \tag{2.20}$$

then $\beta f'_\Lambda - \beta f_\Lambda$, expanded in powers of z, is given by Stirling's series which is Borel summable in the z variable [14] but not in y; (see also Sect. 4).

We now introduce the notation

$$R_n^q(\gamma;y) \equiv \frac{Q_n^q(\gamma;y)}{\prod_{k=q}^{n-1+q}(1+ky)} = \sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\gamma^{-\ell-q}(-1)^{n-\ell}}{\prod_{k=q}^{\ell-1+q}(1+ky)}\,, \tag{2.21}$$

and

$$R_n^q(y) = R_n^q(1;y) , \quad Q_n^q(y) = Q_n^q(1;y) \tag{2.22}$$

Clearly $Q_n^q(\gamma;y)$ is a polynomial in y, and $R_n^q(\gamma;y)$ is a rational function of y which is analytic in y in the entire complex plane, except in the interval $[-1,0]$. In general $Q_n^q(\gamma;y=0) \neq 0$. The point is that for $\gamma = 1$ we have the following result.

Lemma 2.1.

For $\gamma = 1$

$$\lim_{y\to 0} |y^{-E(n)} Q_n^q(1;y)| = \lim_{y\to 0} |y^{-E(n)} Q_n(y)|$$

is finite, where

$$E(n) = \text{integer part of } \frac{n+1}{2} . \qquad \square$$

The proof of Lemma 2.1 is given in the appendix.

From (2.17)-(2.19) and (2.21), (2.22) we obtain the representations

$$Z_3 = \sum_{\Omega} \frac{1}{y^{|\Omega|}|\Omega|!} J_{\Omega} \prod_{j\in\Lambda} R^0_{n_j(\Omega)}(y) , \tag{2.23}$$

and

$$Z_3 \left\langle \phi_x^1 \phi_u^1 \right\rangle_{\beta,z} = \sum_{\substack{\Omega \\ \omega : x \to u}} \frac{1}{y^{|\Omega|}\,|\Omega|!}\, J_\Omega J_\omega \cdot$$

$$\cdot \prod_{j \in \Lambda} R_{n_j(\omega,\Omega)}^{\delta_{xj}+\delta_{uj}}(y) , \qquad (2.24)$$

with $R_n^q(y)$ as in (2.21), (2.22), and

$$n_j(\Omega) = \sum_{\omega' \in \Omega} n_j(\omega') , \quad \text{see (2.12)} ,$$

$$n_j(\omega,\Omega) = n_j(\omega) + \sum_{\omega' \in \Omega} n_j(\omega') ,$$

see (2.14).

It is easy to see that

$$\sum_{j \in \Lambda} E(n_j) > |\Omega| .$$

Therefore the singular factor $y^{-|\Omega|}$ on the r.s. of (2.23) and (2.24) is <u>cancelled</u> by

$$\prod_{j \in \Lambda} Q_{n_j}^q(\gamma;y)$$

provided $\gamma = 1$, i.e. $C_{\beta,jj} = 1$ (see (2.4)), as follows from Lemma 2.1. Thus, we now understand the crucial rôle of condition (1.10).

The proof of analyticity of $\beta f(\beta,y)$ and of $\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z}$ in y in the domain

$$\{y : |\arg y| < \pi-\varepsilon \quad \text{or} \quad |y| > 1+\varepsilon\}$$

for β sufficiently small (depending on ε), and of Borel summability in y at $y = 0$ is now reduced to proving convergence of the expansions (2.23) and (2.24) and of their Taylor remainders (at $y = 0$) in the above domains of y and β, uniformly in $|\Lambda|$.

We shall establish convergence by applying the polymer method [10-12] to the r.s. of (2.23) and (2.24). We first notice that a term on the r.s. of (2.23) or (2.24) indexed by a family $\Omega = \{\omega_1,\ldots,\omega_m\}$ of oriented random loops does not depend on the ordering of $\{\omega_1,\ldots,\omega_m\}$. We may thus resum over all families of oriented random loops which only differ in their ordering. This yields expansions of Z_3 and $\langle \phi_x^1 \phi_u^1 \rangle_{\beta,z}$ with terms labelled by what are called multi-indices in [10], i.e. by functions from the set Γ of all random walks to $\mathbb{N}$. In the following, these labels, the multi-indices are called g-sets (for "generalized sets") of random walks. A g-set may

contain an arbitrary number of copies of the same random walk. We shall use set-theoretical language (whenever it is unambiguous), speaking of unions of g-sets = addition of multi-indices, of elements of g-sets = particular copies of particular random walks in a g-set, etc. Let Ω be some g-set. Given some random walk $\omega \in \Gamma$, let $\nu(\Omega,\omega)$ be the number of copies of ω appearing in Ω. We define

$$[\Omega]! = \prod_{\omega \in \Gamma} \nu(\Omega,\omega)! \tag{2.25}$$

with the usual convention that $0! = 1$.

We may now rewrite expansions (2.23) and (2.24) by summing over all distinct g-sets Ω of oriented random loops (rather than over arbitrary, ordered collections of random loops) and replacing $|\Omega|!$ by $[\Omega]!$.

Next, we observe that each term in these modified expansions labelled by a disconnected g-set can be factorized in a product of terms labelled by connected g-sets, also called polymers, and denoted by Ω^c. (A g-set Ω is called connected, i.e. a polymer, if it is given by oriented random loops, or walks $\omega_1,\ldots,\omega_m$, not necessarily distinct, with the property that $\omega_k \cap \omega_{k+1} \neq \emptyset$, $k = 1,\ldots,m-1$.) Two polymers are said to be compatible if their union is not a polymer. (Otherwise

they are said to be incompatible). If $\tilde{\Omega}$ is a g-set of oriented random loops with the property that $\tilde{\Omega} \cup \{\omega\}$ is a polymer, where $\omega : x \to u$ is an open random walk then $\Omega^c = \tilde{\Omega} \cup \{\omega\}$ is called an $x \to u$ polymer.

Given some polymer, Ω^c, we let $|\Omega^c|$ denote the total number of oriented random loops in Ω^c, counting multiplicities,

$$p(\Omega^c) = \sum_{\omega \in \Omega^c} p(\omega) , \tag{2.26}$$

(we recall that $p(\omega)$ is the number of steps made by the random walk ω),

$$n_j(\Omega^c) = \sum_{\omega \in \Omega^c} n_j(\omega) , \quad j \in \Lambda , \tag{2.27}$$

$$\ell(\Omega^c) = \sum_{\omega \in \Omega^c} \ell(\omega) . \tag{2.28}$$

Clearly

$$2|\Omega^c| \leq p(\Omega^c) = \sum_{j \in \Lambda} n_j(\Omega^c) \leq \ell(\Omega^c) . \tag{2.29}$$

We also define an activity, z, of each polymer,

$$z(\Omega^c) = \frac{1}{y^{|\Omega^c|}[\Omega^c]!} J_{\Omega^c} \prod_{j\in\Lambda} R_{n_j(\Omega^c)}^{q_j}(y) , \qquad (2.30)$$

where $q_j = \delta_{xj} + \delta_{uj}$ if Ω^c is an $x \to u$ polymer, and $q_j = 0$, otherwise. (Note that $|\Omega^c|$ is the total number of random loops in Ω^c, not counting open random walks.)

The expansions (2.23) and (2.24) can now be written as sums of products of activities associated with compatible polymers. The constraint of compatibility is equivalent to a hard core exclusion between polymers. Let

$$g(\Omega^c,\Omega^{c'}) = \begin{cases} 0 & \text{if } \Omega^c \text{ and } \Omega^{c'} \text{ are} \\ & \text{compatible polymers} \\ -1, & \text{otherwise} \end{cases} \qquad (2.31)$$

Then the expansions (2.23) and (2.24) take the form

$$Z_3 = \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c,\ldots,\Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \prod_{1\le k<k'\le r} [1 + g(\Omega_k^c,\Omega_{k'}^c)] \qquad (2.32)$$

$$\langle \phi^1_x \phi^1_u \rangle_{\beta,z} = Z_3^{-1} \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c,\ldots,\Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \cdot$$

$$\cdot \prod_{1\le k<k'\le r} [1 + g(\Omega_k^c, \Omega_{k'}^c)] \qquad (2.33)$$

where in (2.32) summation extends over all sets of polymers made of oriented random loops while in (2.33) exactly one polymer is an $x \to u$ polymer. The factor

$$\prod_{1\le k<k'\le r} [1 + g(\Omega_k^c, \Omega_{k'}^c)]$$

is the Boltzmann factor of the hard core interaction between polymers. The activities $z(\Omega^c) = z(\Omega^c;y)$ are functions of $y = 2/N$.

Criteria for convergence of the expansions (2.32) and (2.33) are well known [11,12]. See, in particular, the clear presentation in [10]. In the appendix we prove the following bound on the activity $z(\Omega^c)$ of a polymer, Ω^c:

Lemma 2.2.

If y is such that $|\arg y| < \pi-\varepsilon$ or $|y| > 1+\varepsilon$ then there exists a finite constant K_ε such that

$$|z(\Omega^c)| \leq [2d\beta K_\varepsilon (m^2(\beta,\Lambda))^{-1}]^{\ell(\Omega^c)} , \tag{2.34}$$

where $m^2(\beta,\Lambda)$ is the solution of Eq. (1.10).

Remarks.

1) It follows from [10-12] and (2.34) that the expansions (2.32) and (2.33) can be exponentiated and converge uniformly in $|\Lambda|$ if β is sufficiently small, ($\beta < \text{const. } K_\varepsilon^{-1}$). In the next section we recall some of the machinery developed in [10-12], following [10], which can be used to prove convergence.

2) The main analytical facts needed to prove (2.34) are the following estimates:

a) For $k \leq E(n)$,

$$|y^{-k} Q_n^q(y)| \leq \text{const}_q{}^n \, k! \prod_{k=q}^{n-1+q} (1 + k|y|) , \tag{2.35}$$

where Q_n^q has been defined in (2.21), (2.22).

b) $$|J_{\Omega^c}| \leq \left[\frac{2d\beta}{m^2(\beta,\Lambda)} \right]^{\ell(\Omega^c)} , \tag{2.36}$$

with $m(\beta,\Lambda) \searrow 1$, as $\beta \nearrow 0$; (see (1.10), (1.11)).

Fact a) follows from the definition of Q_n^q and Lemma 2.1 which asserts that, for $k \leq E(n)$, $y^{-k} Q_n^q(y)$ is analytic near $y = 0$, so that one can use the Cauchy estimate. Fact b) is

standard (see e.g. [6]). Details are provided in the appendix. Lemma 2.2 follows from facts a) and b) combined with some fairly subtle combinatorial arguments contained in the appendix.

3. Convergence of the Expansions (2.32) and (2.33), and Proof of Theorem A

The form of expansions (2.32) and (2.33) and estimate (2.34) on the activity of polymers permit us to use the polymer method described in [10-12] in order to prove convergence. (We follow [10].) We now briefly recall the main features of that method and some combinatorial estimates which we also need in our proof of Lemma 2.2, contained in the appendix.

Our aim is to prove existence of the thermodynamic limit, analyticity properties in y and Borel summability of the 1/N expansion of

$$\beta f_{\Lambda}(\beta,y) = |\Lambda|^{-1} \log Z_3 , \tag{3.1}$$

and

$$\langle \phi^1_x \phi^1_u \rangle_{\beta,z} = \frac{d}{dt} \log \left[Z_3 + t Z_3 \langle \phi^1_x \phi^1_u \rangle_{\beta,z} \right] \Big|_{t=0} \tag{3.2}$$

By (2.33), the unnormalized two point function $tZ_3 \langle \phi^1_x \phi^1_u \rangle_{\beta,z}$, has a polymer expansion in which the activities of the $x \to u$ polymer are multiplied by a factor t. In order to control the logarithms of polymer expansions we require two basic properties (A and B, below) of polymers; (see e.g. [10]).

Property A. Let Ω_0^c be a polymer of length ℓ_0. The total number of polymers, Ω^c, of length ℓ which are incompatible with Ω_c^0 is bounded by $\ell_0 K_1^{\ell}$, where K_1 is a constant proportional to d, and d is the dimension of the lattice.

To verify Property A in our case, we first note that we may associate with each polymer Ω^c a walk $\overline{\Omega^c}$, and two subsets $S(\Omega^c)$ and $E(\Omega^c)$ made of jumps in $\overline{\Omega^c}$. We construct $\overline{\Omega^c}$ first. We start at some site $j_1 \in \Omega^c$ ($j_1 = x$ if Ω^c is an $x \to u$ polymer). All walks visiting j_1 form a g-subset H_1. We choose one of them and follow its jumps till we arrive at some site j_2 visited by a non-empty g-subset H_2 of walks in $\Omega^c \sim H_1$. We choose one of the walks in H_2 and follow its jump till we arrive at a site j_3 visited by some walk in $\Omega^c \sim H_1 \sim H_2$, etc. Sometimes the walk in $H_k, k \geq 1$, that we were following terminates. This can only happen at the site j_k. We then choose another walk in H_k whose jumps we have not yet followed, or -- if no such walk exists anymore -- we follow further the jumps of the walk in H_{k-1} along which we arrived at j_k the first time. In this way we must finally get back to j_1 (or land at u if Ω^c is an $x \to u$ polymer), having completely followed all the paths in Ω^c (provided, if Ω^c is an $x \to u$ polymer, we choose the $x \to u$ path in H_1 as late as possible...). The walk which we have followed in this fashion is $\overline{\Omega^c}$. Our definition of $\overline{\Omega^c}$ depends most of the time of many arbitrary choices, but this need not concern us.

The subsets $S(\Omega^c)$ and $E(\Omega^c)$ are defined by the following condition: a jump j in $\overline{\Omega^c}$ belongs to $S(\Omega^c)$ (respectively: to $E(\Omega^c)$) if and only if there exists a walk ω in Ω^c such that the jump j is the first one (respectively: the last one) that we followed on the walk ω in the process described above.

We claim that given a random walk ω and two subsets S and E of its jumps, there exists at most one polymer Ω^c such that $\overline{\Omega^c} = \omega$, $S(\Omega^c) = S$ and $E(\Omega^c) = E$. Indeed, if such a polymer Ω^c exists, by the way $\overline{\Omega^c}$ is constructed, there has to be a subloop ω_1 of ω, i.e. a closed sequence of consecutive jumps in ω, which start (respectively: end) by a jump in S (respectively: in E) and is minimal with this property. If $\omega^1 = \omega \sim \omega_1$ is not empty, we can find a subloop ω_2 of ω^1 with the same property, replacing S and E by $S_1 = S \cap \omega^1$ and $E_1 = E \cap \omega^1$. Finally we end up with ω^{k+1} empty, and Ω^c must be $\{\omega_1, \omega_2, \ldots, \omega_k\}$. There is no arbitrariness in fact in this construction of Ω^c and this proves the claim. Since there are no more than $4^{p(\omega)}$ possible subsets S and E for a given ω, and since there are no more than $\ell_0(2d)^\ell$ random walks ω of length ℓ incompatible with Ω_0^c the number of polymers, Ω^c, of length ℓ incompatible with Ω_0^c is bounded by

$$\ell_0 \, 4^{p(\Omega^c)} (2d)^\ell \leq \ell_0 (8d)^\ell \tag{3.3}$$

which proves Property A.

Similar arguments show that the total number of polymers $\Omega_1^c, \Omega_2^c, \ldots$, with $\Omega_k^c \cap \Omega_{k+1}^c \neq \emptyset$, $\Omega_1^c \cup \Omega_2^c \cup \ldots$ contains 0, and $\ell = \sum_{k=1,2,\ldots} \ell(\Omega_k^c)$ fixed is bounded by $(16d)^{\ell}$.

Property B. The activity $z(\Omega^c)$ of each polymer Ω^c is bounded by $K_2^{\ell(\Omega^c)}$, where K_2 is some geometric constant depending on d.

Obviously, Property B follows from Lemma 2.2: For $\beta < \beta_\varepsilon < \beta_S$,

$$K_\varepsilon \beta (2d\beta + m^2(\beta,\Lambda))^{-1} < K_2 . \qquad (3.4)$$

We now perform the cluster expansion, following closely [10].

In what follows, g-sets of polymers are denoted by X. If a g-set of polymers, X, contains precisely one $x \to u$ polymer we write $X : x \to u$. With each g-set of polymers, X, we associate a graph G(X) whose vertices are the polymers in X. Two vertices of G(X) are joined by a line iff the corresponding polymers are incompatible. The total number of lines in a graph, C, is denoted by L(C). Given a g-set X of polymers and a polymer Ω^c, let $\nu(X,\Omega^c)$ be the number of copies of Ω^c in X, and let

$$[X]! = \prod_{\Omega^c} \nu(X,\Omega^c)! \tag{3.5}$$

Finally

$$z(X) = \prod_{\Omega^c} z(\Omega^c)^{\nu(X,\Omega^c)} \tag{3.6}$$

We now have

Lemma 3.1, [10,11]

$$\beta f_\Lambda(\beta,y) = |\Lambda|^{-1} \sum_X \frac{1}{[X]!} \phi^T(X) z(X) , \tag{3.7}$$

$$\langle \phi^1_x \phi^1_u \rangle_{\beta,z} = \sum_{X: x \to u} \frac{1}{[X]!} \phi^T(X) z(X) , \tag{3.8}$$

where

$$\phi^T(X) = \sum_{C \subset G(X)} (-1)^{L(C)} ,$$

and the sum ranges over all connected subgraphs C of G(X) containing all the vertices of G(X). □

Note that $\phi^T(X) = 0$, unless G(X) is connected (i.e. $X = \{\Omega^c_1,\ldots,\Omega^c_r\}$, $r = 1,2,3,\ldots$, with Ω^c_k and Ω^c_{k+1} incompatible, for all $k = 1,\ldots,r-1$).

Lemma 3.2, [10,12]

$$|\phi^T(X)| \leq \prod_{\Omega^c \in X} I_X(\Omega^c) , \tag{3.9}$$

where $I_X(\Omega^c)$ is the number of polymers in X which are incompatible with Ω^c, [i.e. $I_X(\Omega^c)$ is the number of lines of G(X) leaving the vertex corresponding to Ω^c]. □

The key combinatorial lemma based on Property A is the following

Lemma 3.3, (Malyshev's theorem [12], [10])

$$\prod_{\Omega \in X} I_X(\Omega^c) \leq [X]! \, K_3^{\ell(X)} , \tag{3.10}$$

where

$$\ell(X) = \sum_{\Omega^c \in X} \ell(\Omega^c) ,$$

and K_3 is a geometric constant depending on d. □

We do not recall the proofs of Lemmas 3.2, 3.3; see [10,12]. We have stated them here, because they are basic in our proof of convergence of the expansions (3.7), (3.8), see Lemma 3.1, and because we also need them to prove the required

estimates on $z(\Omega^c)$, Lemma 2.2; (for this purpose X is replaced by Ω^c and Ω^c by ω, see Lemmas A.1, A.2, Appendix).

Combining (3.6) with Lemmas 3.1 through 3.3 we obtain

Theorem 3.1.

Let y be such that $|\arg y| < \pi-\varepsilon$ or $|y| > 1+\varepsilon$. Let β be so small that

$$K_\varepsilon \beta (2d\beta + m^2(\beta))^{-1} < K_2 , \tag{3.11}$$

where $K_2 \propto d^{-1}$ is the constant introduced in Property B.

Then the cluster expansions (3.7), (3.8) for the free energy and the correlation functions converge uniformly in $|\Lambda|$. The limits, as $\Lambda \to \mathbb{Z}^d$, exist and are analytic in y in the domain defined above. The limiting connected correlation functions have exponential cluster properties.

Remarks.

1) The last part of Theorem 3.1 is a standard consequence of convergence of the cluster expansion.

2) Condition (3.11) can be satisfied for sufficiently small β, because $m(\beta) \to 1$, as $\beta \to 0$; see Conditions (1.10), (1.11). Moreover, if (3.11) holds then it also holds for sufficiently large, finite Λ, because

$$\lim_{\Lambda \to \mathbb{Z}^d} m(\beta,\Lambda) = m(\beta).$$

3) Although we have tacitly assumed that periodic b.c. were imposed at $\partial\Lambda$, our techniques extend to a general class of b.c. and can be used to establish independence of the thermodynamic limit on b.c. This requires solving condition (1.10) [see also (2.3)] for Dirichlet b.c. and introducing random walks terminating in sites in the boundary of Λ which arise from contractions with boundary fields. Although we shall not discuss these generalizations any further we think they show some nice features of our techniques.

By estimating Taylor remainders in y of the activities $z(\Omega^c)$ (which are analytic functions of y outside [-1,0]) and applying Lemmas 3.1-3.3 and [13] we obtain

Theorem 3.2.

For β small enough, the 1/N expansions for the free energy and the correlation functions are Borel summable at $1/N = 0$.

Remarks.

1) The required estimates on the Taylor remainders in y of the activities $z(\Omega^c)$ are proven in the appendix (Lemma A.5).

2) Our main result, Theorem A, is equivalent to Theorems 3.1 and 3.2.

3) We used Nevanlinna-Sokal's theorem on Borel summability for simplicity, but with a little additional work one can modify Lemma A.5 to verify that the stronger hypotheses of Watson's theorem on Borel summability [15] also hold in this case, provided the temperature is high enough.

4. Discussion of Results and Open Problems

1) We wish to add a comment on the difference between the two free energies $f'(\beta,z)$ and $f(\beta,y)$ introduced in (2.19), (2.20): In Theorem 3.2 we have shown that the expansion of $f(\beta,y)$ in powers of y is Borel summable. However, $\beta[f'(\beta,z) - f(\beta,y)]$ has no expansion in powers of y. This quantity is the normalized volume of a sphere over which the classical spin may range. It is thus the free energy for a 0-dimensional model consisting of a single spin (the "toy integral"). Its expansion in powers of z (not of y) is Stirling's series which is Borel summable. Its Borel transform is given by

$$\frac{1}{z}\beta[f'(\beta,z)-f(\beta,y)] = \sum_{r=0}^{\infty} (-1)^r \frac{B_{r+1}}{[2(r+1)]!} (z')^{2r}$$

$$= \frac{i}{2z'} \cot\left(\frac{iz'}{2}\right) - \left(\frac{1}{z'}\right)^2 , \qquad (4.1)$$

where z' is the "Borel variable dual to z", and the B_r are the Bernoulli numbers. It has singularities on the imaginary axis, at $z' = k\pi i$, $k = \pm 1, \pm 2, \pm 3, \ldots$.

It is an open problem to locate the singularities of the Borel transform of the 1/N expansion and to give such singularities a physical interpretation, as expected of the

singularities in the Borel transform of the coupling constant expansion in field theories like $\lambda\phi_4^4$ [16].

2) The techniques we have used to establish our main results [Theorems 3.1 and 3.2] have the following feature which is both, remarkable and annoying: The cluster expansion of Lemma 3.1 converges for all y with $\mathrm{Re}\, y > 0$, and our estimates on the activities $z(\Omega^c)$ are uniform in y, for $\mathrm{Re}\, y > 0$. Thus the constant $\beta_{\frac{\pi}{2}}$ (see Theorem A, Theorem 3.1) is bounded by

$$\min_{0 < N < \infty} \beta_{\mathrm{crit.}}(N)$$

which is well below β_S.

In contrast, Kupiainen [1] has established estimates which are valid for $0 \leq y < y_0$ and $\beta < \beta_{y_0}$, for small values of y_0, with $\lim_{y_0 \to 0} \beta_{y_0} = \beta_S$.

It is conceivable that we could improve our estimates in this direction, i.e. that we could prove convergence of our cluster expansions in domains $0 < \beta < \beta_{y_0}$, $|\arg y| < \frac{\pi}{2}$, $|y| < y_0$, with $\lim_{y_0 \to 0} \beta_{y_0} \approx \beta_S$. [The $y \to 0$ (spherical model) limit exists term by term in our expansion.] The main obstruction preventing us from establishing such a result is that it is not easy to establish optimal bounds on

$$|y^{-k} Q_n^q(y)| \ ; \tag{4.2}$$

where $Q_n^q(y)$ is one of the polynomials defined in (2.21), (2.22) and $k \leq E(n)$.

We believe that with more work one might be able to improve the estimates on $|y^{-k}Q_n^q(y)|$ [see Lemmas A.1-A.4, in the appendix] and eventually prove the above conjecture.

3) It might be worthwhile to try to extend our methods to other models with a good 1/N-expansion, like the $\mathbb{C}P^{N-1}$ models. The first step in this program consists of choosing the right lattice approximation permitting one to use double [high-temperature, 1/N-] expansions. For the $\mathbb{C}P^{N-1}$ models, it is known that certain lattice actions lead to first order transitions in β at $1/N = 0$ [17,18]. Such actions must be rejected in our scheme. We have reasons to believe that there are lattice actions for the $\mathbb{C}P^{N-1}$ models which are not plagued with such pathologies.

4) It would be interesting to extend our methods to analyze other expansions in statistical physics, e.g. the 1/d expansion. Random walk representation should be particularly convenient to analyze the 1/d expansion [since random walks on $\mathbb{Z}^d$ become mutually avoiding and self-avoiding, as $d \to \infty$, thus identifying this limit with mean field theory. Systematic corrections in 1/d appear to be accessible].

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Appendix. Proofs of Lemmas 2.1 and 2.2 and of Theorem 3.2

In this appendix we supply some technical details which we have used in the main text without proof. The key estimates are Lemmas 2.1 and 2.2 and (2.35). In the proof of Lemma 2.2, Lemma 3.3 (Malyshev's theorem [12,10]) plays an important rôle.

A.1 Proof of Lemma 2.1.

Let

$$Q^q_{n,\ell}(y) = \prod_{k=\ell+q}^{n+q-1} (1+ky), \quad Q^q_{n,n}(y) = 1.$$

By (2.21) and (2.22),

$$Q^q_n(y) = \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{n-\ell} Q^q_{n,\ell}$$

We propose to prove

$$Q^q_n(y) = a_k y^k + O(y^{k+1}), \tag{A.1}$$

where $k \geq E(n) \equiv$ integer part of $\frac{n+1}{2}$, and $a_k < \infty$. Clearly, (A.1) yields Lemma 2.1.

Proof of (A.1). Our proof is inductive and makes use of Pascal's triangular formula

$$\binom{n+1}{\ell} = \binom{n}{\ell} + \binom{n}{\ell-1}.$$

To start the induction we note that

$$Q_0^q(y) = 1, \quad \text{and} \quad Q_1^q(y) = -qy, \quad \forall q. \tag{A.2}$$

In order to do the induction step we use the identity

$$Q_{n+1}^q = (n+q)y\left[\sum_{\ell=0}^{n} \binom{n+1}{\ell}(-1)^{n+1-\ell}Q_{n,\ell}^q\right]$$

$$+ \sum_{\ell=0}^{n-1} \binom{n}{\ell}(-1)^{n+1-\ell}\left[Q^q{}_{n,\ell} - Q_{n,\ell+1}^q\right]. \tag{A.3}$$

Now, notice that

$$\binom{n}{\ell}\left[Q_{n,\ell}^q - Q_{n,\ell+1}^q\right] = ny\binom{n-1}{\ell-1}Q_{n,\ell+1}^q + qy\binom{n}{\ell}Q_{n,\ell+1}^q.$$

Inserting this identity into (A.3) we get

$$
\begin{aligned}
Q^q_{n+1} &= ny\left[\sum_{\ell=0}^{n}\binom{n+1}{\ell}(-1)^{n+1-\ell}Q^q_{n,\ell}\right.\\
&\qquad\left. - \sum_{\ell=2}^{n}\binom{n-1}{\ell-2}(-1)^{n+1-\ell}Q^q_{n,\ell}\right]\\
&\quad + qy\left[\sum_{\ell=0}^{n}\binom{n+1}{\ell}(-1)^{n+1-\ell}Q^q_{n,\ell}\right.\\
&\qquad\left. - \sum_{\ell=1}^{n}\binom{n}{\ell-1}(-1)^{n+1-\ell}Q^q_{n,\ell}\right]\\
&= ny\left[2\sum_{\ell=1}^{n}\binom{n-1}{\ell-1}(-1)^{n+1-\ell}Q^q_{n,\ell}\right.\\
&\qquad\left. + \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-1)^{n-1-\ell}Q^q_{n,\ell}\right]\\
&\quad + qy\left[\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{n+1-\ell}Q^q_{n,\ell}\right]\\
&= ny\left[(1+(n-1)y)Q^q_{n-1} - 2Q^{q+1}_{n-1}\right] - qyQ^q_n
\end{aligned}
\tag{A.4}
$$

Formulas (A.4) and (A.2) yield (A.1) by induction in n. This completes the proof of Lemma 2.1.

A.2 Proof of Lemma 2.2.

Let $\Omega^c = \{\omega_1,\ldots,\omega_m\}$ be some polymer. (Different elements of Ω^c may correspond to different copies of the same random walk.) Our purpose is to estimate the activity $z(\Omega^c)$ of Ω^c by proving a bound on $|y^{-k}Q_n^q(y)|$, $k \le E(n)$, and straightening out the combinatorial properties of polymers. We start with the latter.

Lemma A.1.

Consider some g-set Ω consisting of oriented (not necessarily distinct) random loops. With each site $j \in \Lambda \subseteq \mathbb{Z}^d$ one can associate a (possibly empty) g-subset $I_j = I_j(\Omega)$ of Ω, with the properties that

i) each $\omega \in \Omega$ belongs to precisely one I_j;

ii) each $\omega \in I_j$ visits j;

iii) the total number of walks in I_j, $|I_j|$, satisfies

$$|I_j| \le E(n_j(\Omega)), \quad \text{for all} \quad j \in \Lambda.$$

Remark. Formula (2.30) for $z(\Omega^c)$, property (A.1) and Lemma A.1 (with $\Omega = \Omega^c$, or $\Omega = \Omega^c \sim \{\omega\}$, where $\omega : x \to u$) show that $z(\Omega^c)$ is regular at $y = 2/N = 0$.

Proof. Our proof proceeds by induction in $|\Omega|$. For $|\Omega| = 1$, Lemma A.1 is trivial. We may thus assume it is true for $|\Omega| \le m$.

We then show that it holds for $|\Omega| = m + 1$. Since any closed random loop makes at least two steps, we may choose, for any ω_ℓ in Ω, $\ell = 1,\ldots,m+1$, two distinct sites j_ℓ^1 and j_ℓ^2; we then call ω'_ℓ the closed random loop made of only two steps which visits j_ℓ^1 and j_ℓ^2, and $\Omega' = \{\omega'_1,\ldots,\omega'_{m+1}\}$. It is now clear that if Lemma A.1 holds for Ω', it holds for Ω; indeed there is a one to one correspondence between Ω' and Ω which induces a natural correspondence between the g-subsets $I_j(\Omega')$ and $I_j(\Omega)$; and obviously $n_j(\Omega') \leq n_j(\Omega)$ implies $E(n_j(\Omega')) \leq E(n_j(\Omega)) \forall_j \in \Lambda$. Since it also suffices to prove Lemma A.1 for connected g-sets (otherwise we may apply the induction hypothesis to each connected component) we are left with the case of a connected g-set $\Omega = \{\omega_1,\ldots,\omega_{m+1}\}$ of random loops, each of them made of two steps. Then:

- either there exists $j \in \Lambda$ such that $n_j(\Omega)$ is odd. Then j is visited by a walk $\omega^{(j)} \in \Omega$, and:

$$E(n_j(\Omega)) \geq E(n_j(\Omega \sim \{\omega^j\})) + 1$$

We may then assign $\omega^{(j)}$ to $I_j(\Omega)$ and Lemma A.1 follows from the induction hypothesis for $\Omega \sim \{\omega^{(j)}\}$,

- or:

$$n_j(\Omega) \text{ is even for all } j \in \Lambda. \qquad \text{(A.5)}$$

Let $\omega_1 \in \Omega$. Choose a site $j_1 \in \omega_1$, and assign ω_1 to $I_{j_1}(\Omega)$. Let j_2 be the other site visited by ω_1, and $\omega_2 \neq \omega_1$ a loop of Ω visiting j_2. We assign ω_2 to $I_{j_2}(\Omega)$, and define j_3 as the other site visited by ω_2 (possibly $j_3 = j_1$). If j_3 is visited by a path ω_3 in Ω, not equal to ω_1 or ω_2, we assign ω_3 to $I_{j_3}(\Omega)$, and so on. This construction may stop after having chosen j_k and ω_k, $k \geq 2$. This can only be the case if the other site visited by ω_k is $j_{k+1} = j_1$. Indeed from A.5 and the construction we made, $n_{j_{k+1}}(\Omega \sim \{\omega_1, \ldots, \omega_k\})$ should be odd, hence not zero, except if $j_{k+1} = j_1$. In this way we therefore obtain partial g-subsets $K_j = \{\omega_i \in I_j, i \leq k\}$. We complete the construction if $k < m + 1$, by applying the induction hypothesis to $\Omega^k = \Omega \sim \{\omega_1, \ldots, \omega_k\}$; this constructs g-subsets $I_j(\Omega^k)$, and finally we define

$$I_j(\Omega) = K_j \cup I_j(\Omega^k) .$$

It is easy to verify then all items of Lemma A.1, because by our construction:

$$E(n_j(\Omega)) = E(n_j(\Omega^k)) - |K_j| \qquad \square$$

The next (and last) combinatorial result is an easy consequence of Lemma 3.3.

Lemma A.2.

There is a constant K_4 such that for any closed polymer Ω^c

$$\prod_{j \in \Lambda} |I_j(\Omega^c)|! \leq [\Omega^c]! \, K_4^{\ell(\Omega)} \tag{A.9}$$

where $I_j(\Omega^c)$ is defined as in Lemma A.1, unless Ω^c is an $x \to u$ polymer containing $\omega : x \to u$, in which case $I_j(\Omega^c) = I_j(\Omega^c \sim \{\omega\})$.

Proof. We apply Lemma 3.3, with X replaced by Ω^c and $\Omega^c \in X$ replaced by $\omega \in \Omega^c$. Clearly, Property A holds if we identify Ω_0^c with a random walk ω_0 of length ℓ_0 and count the total number of random walks ω of length $\ell(\omega) = \ell$ incompatible with ω_0. This guarantees that the hypotheses of Lemma 3.3, with $X \to \Omega^c$, $\Omega^c \to \omega$, are fulfilled. Thus, by (3.10),

$$\prod_{\omega \in \Omega^c} I_{\Omega^c}(\omega) \leq [\Omega^c]! K_3^{\ell(\Omega^c)} . \tag{A.10}$$

Now, for $\omega \in I_j(\Omega^c)$, $I_{\Omega^c}(\omega) \geq |I_j|$, with $I_j = I_j(\Omega^c)$. Thus

$$\prod_{\omega \in I_j} I_{\Omega^c}(\omega) \geq |I_j|^{|I_j|} \geq c^{|I_j|} |I_j|! \tag{A.11}$$

Since $$\prod_{j \in \Lambda} \prod_{\omega \in I_j} I_{\Omega^c}(\omega) = \prod_{\omega \in \Omega^c} I_{\Omega^c}(\omega) ,$$

(A.9) follows from (A.10) and (A.11).

End of proof.

We now come to the more analytical part of the proof of Lemma 2.2.

Lemma A.3.

$$0 \leq C_{\beta,ij} \leq \left(\frac{2d\beta}{m(\beta,\Lambda)^2} \right)^{|i-j|} ,$$

$$0 \leq J_{\Omega^c} \leq \left(\frac{2d\beta}{m(\beta,\Lambda)^2} \right)^{\ell(\Omega^c)} .$$

Proof. When $i = j$, $C_{\beta,ij} = 1$, by condition (1.10), so the lemma holds trivially. For $i \neq j$, we apply the random walk expansion of C_β; see [7].

$$C_{\beta,ij} = \sum_{\tilde{\omega} : i \to j} \prod_{k \in \Lambda} \left(\frac{\beta}{2d\beta + m(\beta,\Lambda)^2} \right)^{\tilde{n}_k(\tilde{\omega})} > 0 ,$$

where $\tilde{\omega}$ is a nearest neighbor random walk, and $\tilde{n}_k(\tilde{\omega})$ the number

of visits of $\tilde{\omega}$ at k. There are at most $(2d)^{\ell}$ such walks of length ℓ. Thus

$$C_{\beta,ij} \leq \frac{\beta}{2d\beta+m(\beta,\Lambda)^2} \sum_{\ell=|i-j|}^{\infty} \left(\frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2} \right)^{\ell}$$

$$\leq \left(\frac{\beta}{2d\beta+m^2(\beta,\Lambda)} \right) \left(\frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2} \right)^{|i-j|} \left(1 - \frac{2d\beta}{2d\beta+m(\beta,\Lambda)^2} \right)^{-1}$$

$$\leq \left(\frac{2d\beta}{m(\beta,\Lambda)^2} \right)^{|i-j|} .$$

The bound on J_{Ω^c} follows from the bound on $C_{\beta,ij}$ and the definition of J_{Ω^c}. □

Next, we prove our crucial bounds on the polynomials $Q_n^q(y)$; see (2.35).

Lemma A.4.

For $k \leq E(n)$,

$$|y^{-k} Q_n^q(y)| \leq e^{2(n+q-1)} 2^n k! \prod_{k=q}^{n+q-1} (1+k|y|)$$

Proof.

If $|y| \geq (2\rho)^{-1}$ then, by (2.21) and (2.22)

$$|y^{-k}Q_n^q(y)| \leq (2\rho)^k \left| \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{n-\ell} \prod_{k=\ell+q}^{n+q-1} (1+ky) \right|$$

$$\leq e^{2\rho} k! 2^n \prod_{k=q}^{n+q-1} (1+k|y|)$$

If $|y| \leq (2\rho)^{-1}$ we may apply Cauchy's estimate: We choose Γ to be the circle of radius ρ^{-1} centered at the origin. Let

$$f(y) = y^{-k}Q_n^q(y).$$

By (A.1), $f(y)$ is analytic in $y \in \mathbb{C} \sim \{\infty\}$. For $|y| < (2\rho)^{-1}$, we may therefore apply the Cauchy formula

$$f(y) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{\zeta - y} d\zeta$$

which yields

$$|f(y)| \leq 2\rho^k \max_{\zeta \in \Gamma} \left| \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{n-\ell} \prod_{k=\ell+q}^{n+q-1} (1+k\zeta) \right|$$

$$\leq 2e^{\rho} k! 2^n \prod_{k=q}^{n+q-1} (1+k/\rho).$$

Lemma A.4 now follows by choosing $\rho = n+q-1$. □

Corollary A.5.

For $k \le E(n)$ and y in the domain

$$D_\varepsilon = \{y : |\arg y| < \pi - \varepsilon, \text{ or } |y| > 1+\varepsilon\} \tag{A.12}$$

$$|y^{-k} R_n^q(y)| \le C_\varepsilon^{n+q} k! , \tag{A.13}$$

where R_n^q is the rational function defined in (2.21) and (2.22), and C_ε is a finite constant, for each $\varepsilon > 0$. If $\mathrm{Re}\, y > 0$

$$|y^{-k} R_n^q(y)| \le \left(\sqrt{2}\, 2\right)^n e^{2(n+q-1)} k! \tag{A.14}$$

Proof.

For $y \in D_\varepsilon$, $\prod_{k=q}^{n+q-1} (1+k|y|)|1+ky|^{-1} \le c_\varepsilon^{n+q-1}$, (A.15)

for some constant c_ε which is finite for all $\varepsilon > 0$. If $\mathrm{Re}\, y \ge 0$, $c_\varepsilon \le \sqrt{2}$ in (A.15). Thus Corollary A.5 follows from the definition of R_n^q and Lemma A.1. □

Proof of Lemma 2.2.

By formula (2.30)

$$z(\Omega^c) = \frac{1}{[\Omega^c]!} J_{\Omega^c} y^{-|\Omega^c|} \prod_{j \in \Lambda} R^{q_j}_{n_j(\Omega^c)}(y)$$

$$= \frac{1}{[\Omega^c]!} J_{\Omega^c} \prod_{j \in \Lambda} y^{-|I_j(\Omega^c)|} R^{q_j}_{n_j(\Omega^c)}(y) , \tag{A.16}$$

where $I_j(\Omega^c)$ is as in Lemma A.2, and the last equation follows from Lemma A.1. By Lemma A.1, iii)

$$|I_j(\Omega^c)| \leq E(n_j(\Omega^c)) , \text{ for all } j.$$

We may therefore apply Corollary A.5 to estimate $|y^{-|I_j(\Omega^c)|} R^{q_j}_{n_j(\Omega^c)}(y)|$ which yields:

$$|z(\Omega^c)| \leq \frac{1}{[\Omega^c]!} J_{\Omega^c} \prod_{j \in \Lambda} |I_j(\Omega^c)|! C_\varepsilon^{n_j(\Omega^c)+q_j} \tag{A.17}$$

for $y \in D_\varepsilon$ (defined in (A.12)). If we now use inequality (A.9), Lemma A.2, and the inequality

$$\sum_j n_j(\Omega^c) + q_j \leq \ell(\Omega^c) + 1 \tag{A.18}$$

we find

$$|z(\Omega^c)| \leq J_{\Omega^c}(C_\varepsilon K_4)^{\ell(\Omega^c)} \tag{A.19}$$

which, together with Lemma A.3, completes the proof of Lemma 2 [If Re y $\geq$ 0 we may use (A.14) instead of (A.13).]

A.3 Proof of Theorem 3.2.

It sufficies to prove uniform estimates on the r^{th} derivative in y of $z(\Omega^c;y) \equiv z(\Omega^c)$, for Re y $\geq$ 0 and arbitrary polymers Ω^c. One then uses Lemmas 3.1 through 3.3 to transfer such estimates to $\beta f(\beta,y)$ and $\langle \phi^1_x \phi^1_u \rangle_{\beta,z}$. Therefore we only need one further lemma, namely:

Lemma A.5.

If Re 1/y > 1:

$$\left| \frac{\partial^r}{\partial y^r} z(\Omega^c;y) \right| \leq c\, 2^r (r!)^2 J_{\Omega^c} K_5^{\ell(\Omega^c)} \tag{A.20}$$

for some constants c and K_5.

Proof: According to (A.16) and Leibniz' rule:

$$\frac{\partial^r}{\partial y^r} z(\Omega^c;y) = \frac{1}{[\Omega^c]!} J_{\Omega^c} \sum_{\sum_j s_j = r} \prod_j \frac{\partial^{s_j}}{\partial y^{s_j}} f_j \tag{A.21}$$

where f_j stands for $y^{-|I_j(\Omega^c)|}\, R^{q_j}_{n_j(\Omega^c)}(y)$.

We estimate the derivatives on the right side by using the Cauchy formula, as in the proof of Lemma A.4. Let C_1 be the disk $\{y; \mathrm{Re}\, 1/y > 1\}$ and Γ_ρ the circle surrounding it at distance $(2\rho)^{-1}$, namely the set of points y with $\mathrm{dist.}\{y, C_1\} = (2\rho)^{-1}$.

For f analytic inside Γ_ρ and y in C_1 we have:

$$\frac{\partial^s}{\partial y^s} f(y) = \frac{s!}{2\pi i} \oint_{\Gamma_\rho} \frac{f(\zeta)}{(\zeta - y)^{s+1}}\, d\zeta \qquad (A.22)$$

We can apply this Cauchy formula to the derivative of f_j in the right side of (A.21), provided we choose $\rho = n_j + q_j$; indeed by (A.1), f_j is analytic inside $\Gamma_{n_j+q_j}$.

We need the following easy generalization of (A.14):

$$\left| \prod_{k=q}^{n+q-1} \left[(1+k|y|)(1+ky)^{-1} \right] \right| \leq 4^{n+q} \quad \text{if} \quad y \in \Gamma_{n+q} \qquad (A.23)$$

Using Lemma A.4, it yields for $y \in \Gamma_{n_j+q_j}$:

$$|f_j(y)| \leq 8^{n_j+q_j} e^{2(n_j+q_j-1)} |I_j(\Omega^c)|! \qquad (A.24)$$

We insert this bound in (A.22) and obtain, for y in C_1:

$$\left| \frac{\partial^{s_j}}{\partial y^{s_j}} f_j(y) \right| \leq s_j! (8e^2)^{n_j+q_j} [2(n_j+q_j)]^{s_j+1} |I_j(\Omega^c)|! \quad \text{(A.25)}$$

We use the estimate:

$$[2(n_j+q_j)]^{s_j+1} \leq 2(n_j+q_j) s_j! e^{2(n_j+q_j)} \leq s_j! (2e^2)^{n_j+q_j} \quad \text{(A.26)}$$

We can now bound the sum in the right side of (A.21), using (A.25), (A.26) and (A.18). Since there are at most $2^{r+\ell(\Omega^c)}$ possible choices of sequences $(s_j)_{j \in \Lambda}$ such that $\sum_j s_j = r$ and $s_j = 0$ if $n_j(\Omega^c) = 0$, and since $\prod_j s_j! \leq r!$ we get the bound

$$(r!)^2 \, 2^{r+\ell(\Omega^c)} (16e^4)^{\ell(\Omega^c)+1} \prod_j |I_j(\Omega^c)|!$$

Lemma A.5 follows from this bound and Lemma A.2, with $K_5 = 32e^4 K_4$. Of course we do not try to find best possible constants. Theorem 3.2 now follows by appealing to the result of Nevanlinna and Sokal [13]:

Lemma A.6.

Let f be analytic in the circle $C_R = \{y; \operatorname{Re} 1/y > 1/R\}$. Suppose f admits an asymptotic expansion such that:

$$f(y) = \sum_{k=0}^{r-1} a_k y^k + R_r(y) \tag{A.27}$$

with

$$|R_r(y)| \leq c\sigma^r r! \, |y|^r \tag{A.28}$$

uniformly in r and $y \in C_R$, for some constants σ and c. Then f is Borel summable, which means that the power series $\sum_k a_k t^k/k!$ converges for $|t| < 1/\sigma$, that it defines a function B(t) which has an analytic continuation in the strip

$$S_\sigma = \{t ; \operatorname{dist.}\{t, R_+\} < 1/\sigma\}$$

and that this function satisfies the bound:

$$|B(t)| \leq K \exp t/R \quad \text{for} \quad t \in R_+ .$$

Finally f is represented by the following absolutely convergent integral:

$$f(y) = \frac{1}{y} \int_0^\infty \exp(-t/y) B(t) dt , \quad y \in C_R$$

There is also a reciprocal theorem which we do not use here. We apply this theorem in our case, with $R = 1$, by simply noticing that (A.27) and (A.28) follow from Lemma A.5 by Taylor's formula.

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